Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 37, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EIGENVALUE PROBLEMS FOR SINGULAR MULTI-POINT DYNAMIC EQUATIONS ON TIME SCALES

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Communicated by Mokhtar Kirane

Abstract. In this article, we study a singular multi-point dynamic eigenvalue problem on time scales. We find existence of positive solutions by constructing the Green's function and studying its positivity eigenvalue intervals. Two examples are given to illustrate our results.

1. Introduction

In this article, we consider the following singular m-point dynamic eigenvalue problem on time scales

$$(p(t)u^{\Delta}(t))^{\nabla} + \lambda f(t,u(t)) = 0, \quad t \in (0,1] \cap \mathbb{T}, \tag{1.1}$$

$$\alpha u(0) - \beta p(0)u^{\Delta}(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \gamma u(1) + \delta p(1)u^{\Delta}(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{1.2}$$

Some basic definitions on dynamical systems on time scales can be found in [5, 6]. Throughout this paper, it is assumed that

(H1) $p : (0,1)_{\mathbb{T}} \to (0,+\infty)$ and $\int_0^1 \frac{1}{p(s)}\Delta s$ exists; we let $Q(t) := \int_0^t \frac{1}{p(s)}\Delta s$;

(H2) $\xi_i \in (0,1)_{\mathbb{T}}$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $a_i, b_i \in [0,+\infty)$ with $0 < \sum_{i=1}^{m-2} a_i < \alpha$, $0 < \sum_{i=1}^{m-2} b_i < 1$, $\alpha,\beta,\delta \geq 0, \gamma \leq 0$ and

$$0 < \sum_{i=1}^{m-2} b_i Q(\xi_i) - \gamma Q(1) - \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\gamma - \sum_{i=1}^{m-2} b_i\Big)\Big(\beta + \sum_{i=1}^{m-2} b_i Q(\xi_i)\Big) < \delta;$$

(H3) $f : (0,1)_{\mathbb{T}} \times (0,+\infty) \to [0,+\infty)$ is a continuous function and

$$0 < \int_0^1 Q(s)f(s,w)\nabla s < +\infty.$$

Recently, some authors have proved the existence of positive solutions to boundary value problems on time scales; see for example [1, 2, 4, 9, 10, 11, 12, 13, 15, 17, 21, 22, 23, 24, 25, 30] and the references therein. However, very little work has been

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2010 *Mathematics Subject Classification.* 34B15, 34B16, 34B18, 39A10.
Key words and phrases. Time scales; singular multi-point dynamic equations; eigenvalue problems; Green's function; positive solutions; fixed point index theory.

Submitted March 18, 2016. Published February 2, 2017.

done on the existence of positive solutions of singular dynamic boundary value problem on time scales [7, 8, 18, 19]. Other related results on singular ordinary differential equations and singular difference equations appear in [3, 16, 20, 26, 27, 28, 29].

We would like to mention the following results. DaCunha et al. [8] proved the existence results for the singular three point boundary value problem on time scales

$$\begin{gathered} y^{\Delta\Delta}+f(x,y)=0, \quad x\in(0,1]_{\mathbb{T}}, \\ y(0)=0, \quad y(p)=y(\sigma^2(1)), \end{gathered}$$

where $p\in(0,1)\cap\mathbb{T}$ is fixed and $f(x,y)$ is singular at $y=0$ and possibly at $x=0$, $y=\infty$. Liang et al. [18] considered the singular two point dynamic eigenvalue problem on time scales

$$\begin{gathered} [\rho(t)x^{\Delta}(t)]^{\Delta}+\lambda m(t)f(tx(\sigma(t)))=0, \quad t\in[a,b]_{\mathbb{T}}, \\ \alpha x(a)-\beta x^{\Delta}(a)=0, \quad \gamma x(\sigma(b))+\delta x^{\Delta}(\sigma(b))=0, \end{gathered}$$

where $\rho(t)>0$ on $[a,\sigma(b)]$, such that both the delta derivative of $\rho(t)$ and the integral $\int_a^{\rho(b)}(1/\rho(\tau))\Delta\tau$ exist, $m(\cdot)$ and $f(\cdot,\cdot)$ are given functions, $\alpha,\beta,\gamma,\delta\geq 0$, such that

$$d:=\frac{\gamma\beta}{\rho(a)}+\frac{\alpha\delta}{\rho(\rho(b))}+\alpha\gamma\int_a^{\rho(b)}\frac{1}{\rho(\tau)}\Delta\tau>0.$$

Zhang and Wang [29] considered the existence and multiplicity of positive solutions to singular multi-point boundary value problem

$$\begin{gathered} -(p(t)u'(t))'+F(t,u(t))=0, \quad 0<t<1, \\ u(0)=\sum_{j=1}^m a_ju(x_j), \quad w(1)=\sum_{j=1}^m b_jw(x_j), \end{gathered}$$

where $w(t):=p(t)u'(t), a_jb_j\in[0,+\infty)$ with $0<\sum_{j=1}^m a_j<1$ and $\sum_{j=1}^m b_j<1, x_j\in(0,1)$ with $0<x_1<x_2<\dots<x_m<1$, under certain conditions on p and F. The arguments were based upon the positivity of the Green's function and Krasnosel'skii fixed point theorem.

Motivated by [8, 18, 29, 19], in this article, we study the existence of positive solutions for a singular multi-point dynamic eigenvalue problem on time scales. We allow $f(t,w)$ to be singular at $t=0$ and $w=0$. We find eigenvalue intervals in which there exists at least one positive solution of problem (1.1)-(1.2) by making use of the fixed point index theory. The construction of a new Green's function and its positivity are important to our discussion.

This article is organized as follows. In Section 2, we construct the Green's function and give some lemmas based on the positivity of the Green's function. In Section 3, we find eigenvalue intervals in which there exists at least one positive solution of problem (1.1)-(1.2). In Section 4, we study the existence of positive solutions to boundary value problem (1.1)-(1.2) with $\lambda=1$. Finally, in section 5, we give two examples to illustrate our existence theorems.

2. Green's function and some lemmas

Lemma 2.1. *Let $h:(0,1)_{\mathbb{T}}\to[0,+\infty)$ be continuous and satisfy*

$$0<\int_0^1 Q(s)h(s)\nabla s<+\infty.$$

Let

$$Q(t) := \int_0^t \frac{1}{p(s)}\Delta s, \quad y(t) := Q(t)\int_t^1 h(s)\nabla s, \quad t \in [0,1]_{\mathbb{T}},$$

where the function $p(t)$ *is a nonnegative measurable on* $(0,1]_{\mathbb{T}}$. *Then* $y(0) := \lim_{t\to 0^+} y(t) = 0$.

Proof. If $\int_0^1 h(s)\nabla s < +\infty$, then the lemma is clearly true. We now assume that $\int_0^1 h(s)\nabla s = +\infty$. In this case, the function $y(t)$ can be written in the form

$$\begin{aligned} y(t) &= \int_t^1 Q(s)h(s)\nabla s - \int_t^1 \big(Q(s) - Q(t)\big)h(s)\nabla s \\ &= \int_0^1 H(s-t)Q(s)h(s)\nabla s - \int_0^1 H(s-t)\big(Q(s)-Q(t)\big)h(s)\nabla s \end{aligned}$$

for all $t \in (0,1]_{\mathbb{T}}$, where $H(t)$ is the Heaviside function, i.e., $H(s) = 1$, for $s > 0$ and $H(s) = 0$, for $s \le 0$. Let

$$f_n(s) = H(s-t_n)Q(s)h(s), \quad g_n(s) = H(s-t_n)(Q(s) - Q(t_n))h(s), \quad s \in [0,1]_{\mathbb{T}},$$

where t_n is an arbitrary decreasing sequence that approaches 0 as $n \to \infty$. Then

$$\lim_{n\to\infty} f_n(s) = \lim_{n\to\infty} g_n(s) = Q(s)h(s).$$

Applying the Levi monotone convergence theorem or the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} y(0) = \lim_{t\to 0^+} y(t) &= \lim_{n\to\infty}\int_0^1 f_n(s)\nabla s - \lim_{n\to\infty}\int_0^1 g_n(s)\nabla s \\ &= \int_0^1 \lim_{n\to\infty} f_n(s)\nabla s - \int_0^1 \lim_{n\to\infty} g_n(s)\nabla s \\ &= \int_0^1 Q(s)h(s)\nabla s - \int_0^1 Q(s)h(s)\nabla s = 0. \end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let the assumptions of Lemma 2.1 be satisfied. Then the boundary-value problem*

$$(p(t)u^{\Delta}(t))^{\nabla} + h(t) = 0, \quad t \in (0,1]_{\mathbb{T}}, \tag{2.1}$$

$$\alpha u(0) - \beta p(0)u^{\Delta}(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \gamma u(1) + \delta p(1)u^{\Delta}(1) = \sum_{i=1}^{m-2} b_i u(\xi_i) \tag{2.2}$$

has a unique solution given by

$$u(t) = \int_0^1 G(t,s)h(s)\nabla s, \quad t \in [0,1]_{\mathbb{T}}. \tag{2.3}$$

Here the Green's function is defined by

$$\begin{aligned} G(t,s) := D(t,s) + \frac{1}{d}\Big[&\frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} b_i - \gamma\Big)\Big(\sum_{i=1}^{m-2} a_i D(\xi_i,s) + \beta\Big) \\ &+ \sum_{i=1}^{m-2} b_i D(\xi_i,s) - \gamma Q(s)\Big]\Big[\frac{\beta + \sum_{i=1}^{m-2} a_i Q(\xi_i)}{\alpha - \sum_{i=1}^{m-2} a_i} + Q(t)\Big] \end{aligned}$$

$$+\frac{1}{\alpha-\sum_{i=1}^{m-2}a_i}\Big(\sum_{i=1}^{m-2}a_iD(\xi_i,s)+\beta\Big),$$

where

$$D(t,s):=\min\{Q(t),Q(s)\},$$
$$d=\gamma Q(1)+\delta-\sum_{i=1}^{m-2}b_iQ(\xi_i)+\frac{1}{\alpha-\sum_{i=1}^{m-2}a_i}\Big(\gamma-\sum_{i=1}^{m-2}b_i\Big)\Big(\beta+\sum_{i=1}^{m-2}a_iQ(\xi_i)\Big).$$

Proof. We suppose that C_1 and C_2 are arbitrary constants. Let

$$\begin{aligned} u(t)&:=\int_0^t Q(s)h(s)\nabla s+\int_t^1 Q(t)h(s)\nabla s+C_1Q(t)+C_2\\ &=\int_0^1 D(t,s)h(s)\nabla s+C_1Q(t)+C_2,\quad t\in[0,1]_{\mathbb{T}}. \end{aligned}\tag{2.4}$$

Then we have

$$\begin{aligned} u^\Delta(t)&=\Big(\int_0^t Q(s)h(s)\nabla s+\int_t^1 Q(t)h(s)\nabla s+C_1Q(t)+C_2\Big)^\Delta\\ &=\Big(\int_0^t Q(s)h(s)\nabla s+C_2\Big)^\Delta+\Big(Q(t)\int_t^1 h(s)\nabla s\Big)^\Delta+C_1Q^\Delta(t)\\ &=\Big(\int_0^t Q(s)h(s)\nabla s\Big)^\Delta+Q(\sigma(t))\Big(\int_t^1 h(s)\nabla s\Big)^\Delta\\ &\quad+Q^\Delta(t)\int_t^1 h(s)\nabla s+\frac{C_1}{p(t)}\\ &=Q(\sigma(t))h(\sigma(t))-Q(\sigma(t))h(\sigma(t))\\ &\quad+\frac{1}{p(t)}\int_t^1 h(s)\nabla s+\frac{C_1}{p(t)},\quad t\in(0,1]_{\mathbb{T}}\\ &=\frac{1}{p(t)}\int_t^1 h(s)\nabla s+\frac{C_1}{p(t)},\quad t\in(0,1]_{\mathbb{T}}, \end{aligned}\tag{2.5}$$

where we used the delta derivative product rule, and

$$p(t)u^\Delta(t)=\int_t^1 h(s)\nabla s+C_1,\quad t\in(0,1]_{\mathbb{T}}.\tag{2.6}$$

Hence

$$\begin{aligned} (p(t)u^\Delta(t))^\nabla&=\Big(\int_t^1 h(s)\nabla s+C_1\Big)^\nabla\\ &=\Big(-\int_1^t h(s)\nabla s\Big)^\nabla=-h(t),\quad t\in(0,1]_{\mathbb{T}}, \end{aligned}\tag{2.7}$$

which shows that the function $u(t)$ defined by (2.4) is a general solution of (2.7).

We are going to find a solution to problem (2.1) and (2.2). From (2.2), (2.4)-(2.6) and (H2), we find the system of two equations

$$\alpha C_2-\beta\int_0^1 h(s)\nabla s-\beta C_1$$

$$
\begin{aligned}
&= C_2\sum_{i=1}^{m-2} a_i + C_1\sum_{i=1}^{m-2} a_iQ(\xi_i) + \sum_{i=1}^{m-2} a_i\int_0^1 D(\xi_i,s)h(s)\nabla s,\\
&\gamma C_2 + \gamma C_1 Q(1) + \gamma\int_0^1 Q(s)h(s)\nabla s + \delta C_1\\
&= C_2\sum_{i=1}^{m-2} b_i + C_1\sum_{i=1}^{m-2} b_iQ(\xi_i) + \sum_{i=1}^{m-2} b_i\int_0^1 D(\xi_i,s)h(s)\nabla s.
\end{aligned}
$$

Rearranging these equations, we have

$$
\begin{aligned}
&\Big(\alpha - \sum_{i=1}^{m-2} a_i\Big)C_2\\
&= C_1\Big(\beta + \sum_{i=1}^{m-2} a_iQ(\xi_i)\Big) + \sum_{i=1}^{m-2} a_i\int_0^1 D(\xi_i,s)h(s)\nabla s + \beta\int_0^1 h(s)\nabla s,\\
&\Big(\gamma Q(1) + \delta - \sum_{i=1}^{m-2} b_iQ(\xi_i)\Big)C_1\\
&= \sum_{i=1}^{m-2} b_i\int_0^1 D(\xi_i,s)h(s)\nabla s + C_2\sum_{i=1}^{m-2} b_i - \gamma C_2 - \gamma\int_0^1 Q(s)h(s)\nabla s.
\end{aligned}
$$

Solving for C_1 and C_2 yields

$$
\begin{aligned}
C_1 = \int_0^1 \frac{1}{d}\Big\{&\frac{\sum_{i=1}^{m-2} b_i}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} a_iD(\xi_i,s) + \beta\Big) + \sum_{i=1}^{m-2} b_iD(\xi_i,s)\\
&- \frac{\gamma}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} a_iD(\xi_i,s) + \beta\Big) - \gamma Q(s)\Big\}h(s)\nabla s
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 = \int_0^1 \Big\{&\frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\frac{1}{d}\Big\{\frac{\sum_{i=1}^{m-2} b_i}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} a_iD(\xi_i,s) + \beta\Big) + \sum_{i=1}^{m-2} b_iD(\xi_i,s)\\
&- \frac{\gamma}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} a_iD(\xi_i,s) + \beta\Big) - \gamma Q(s)\Big\}\Big)\Big(\beta + \sum_{i=1}^{m-2} a_iQ(\xi_i)\Big)\\
&+ \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} a_iD(\xi_i,s) + \beta\Big)\Big\}h(s)\nabla s.
\end{aligned}
$$

Substituting C_1 and C_2 in (2.4), we find that

$$
u(t) = \int_0^1 G(t,s)h(s)\nabla s, \quad t \in [0,1]_{\mathbb{T}},
$$

where $G(t,s)$ is defined as in Lemma 2.2. Clearly, the constants C_1 and C_2 are uniquely determined by the boundary conditions, the function u is a unique solution to problem (2.1) and (2.2). □

We discuss the positivity of $G(t,s)$. It is clear that

$$
G(t,s) > 0, \quad (t,s) \in [0,1]_{\mathbb{T}} \times (0,1]_{\mathbb{T}}. \tag{2.8}
$$

Lemma 2.3. *The unique solution u of problem* (2.1) *and* (2.2) *satisfies*

$$\eta u(1) \le u(0) \le u(t) \le u(1), \quad t \in [0,1]_{\mathbb{T}},$$

where

$$\eta := \inf\Big\{\frac{G(0,s)}{G(1,s)} : \quad s \in (0,1]_{\mathbb{T}}\Big\} > 0.$$

Proof. From Lemma 2.2 we know that

$$u(t) = \int_0^1 G(t,s)h(s)\nabla s, \quad t \in [0,1]_{\mathbb{T}},$$

where $G(t,s)$ is defined as in Lemma 2.2 and satisfies (2.8). From (2.5), we know that

$$u^{\Delta}(t) = \frac{1}{p(t)}\int_t^1 h(s)\nabla s + \frac{C_1}{p(t)} \ge 0, \quad t \in (0,1]_{\mathbb{T}},$$

therefore $u(0) \le u(t) \le u(1)$ on $[0,1]_{\mathbb{T}}$. Note that

$$\begin{aligned} G(0,s) = \frac{1}{d}\Big(&\frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} b_i - \gamma\Big)\Big(\sum_{i=1}^{m-2} a_i D(\xi_i,s) + \beta\Big) \\ &+ \sum_{i=1}^{m-2} b_i D(\xi_i,s) - \gamma Q(s)\Big)\Big(\frac{\beta + \sum_{i=1}^{m-2} a_i Q(\xi_i)}{\alpha - \sum_{i=1}^{m-2} a_i}\Big) \\ &+ \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} a_i D(\xi_i,s) + \beta\Big), \end{aligned}$$

and

$$\begin{aligned} G(1,s) = Q(s) + \frac{1}{d}\Big(&\frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} b_i - \gamma\Big)\Big(\sum_{i=1}^{m-2} a_i D(\xi_i,s) + \beta\Big) \\ &+ \sum_{i=1}^{m-2} b_i D(\xi_i,s) - \gamma Q(s)\Big)\Big(\frac{\beta + \sum_{i=1}^{m-2} a_i Q(\xi_i)}{\alpha - \sum_{i=1}^{m-2} a_i} + Q(1)\Big) \\ &+ \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\Big(\sum_{i=1}^{m-2} a_i D(\xi_i,s) + \beta\Big). \end{aligned}$$

It is clear that $G(0,s) < G(1,s)$, $s \in (0,1]_{\mathbb{T}}$. We obtain

$$0 < \eta \le \frac{G(0,s)}{G(1,s)} < 1,$$

where

$$\eta := \inf\Big\{\frac{G(0,s)}{G(1,s)} : \quad s \in (0,1]_{\mathbb{T}}\Big\}.$$

As a result, we have

$$\eta u(1) = \int_0^1 \eta G(1,s)h(s)\nabla s \le \int_0^1 G(0,s)h(s)\nabla s = u(0).$$

□

Let $E = C[0,1]_{\mathbb{T}}$ be a Banach space equipped with the supremum norm and

$$P := \{u \in E : \eta\|u\| \le u(t), \quad t \in [0,1]_{\mathbb{T}}\},$$

where $\eta > 0$ is given by Lemma 2.3. Then P is a cone in E. For $u \in P$, we define

$$(Au)(t) = \lambda \int_0^1 G(t,s) f^*(s,u(s)) \nabla s, \quad \forall u \in P, \quad 0 < \lambda, \tag{2.9}$$

where

$$f^*(t,w) := f(t, \max\{w, \eta\tau\}) \tag{2.10}$$

Here τ is a small positive number to be determined. Note that f^* has effectively "removed the singularity" in $f(t,w)$ at $w = 0$, therefore Au is well defined.

By Lemma 2.3, we have for each fixed $u \in P$,

$$\|Au\| = (Au)(1) = \lambda \int_0^1 G(1,s) f^*(s,u(s)) \nabla s, \tag{2.11}$$

$$\eta\|Au\| \le \lambda \int_0^1 G(0,s) f^*(s,u(s)) \nabla s = (Au)(0) \le (Au)(t), \quad \forall t \in [0,1]_{\mathbb{T}}. \tag{2.12}$$

Therefore, $A : P \to P$. Additionally, it is easy to check that A is a completely continuous mapping.

Theorem 2.4 ([14]). *Let P be a cone in a Banach space $E, \Omega \subset E$ a bounded open set, $0 \in \Omega$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator.*

- (A1) *If $Ax = \mu x$, $x \in P \cap \partial\Omega \Rightarrow \mu < 1$, then the fixed point index $i(A, P \cap \Omega, P) = 1$.*
- (A2) *If there exists $y \in P$, $y \ne 0$, such that $x - Ax \ne vy$, for all $x \in P \cap \partial\Omega$, $0 \le v$, then the fixed point index $i(A, P \cap \Omega, P) = 0$.*

3. Positive solutions to eigenvalue problems (1.1)-(1.2)

For an arbitrary constant r, we let $\Omega_r = \{u \in E : \|u\| < r\}$. Then

$$\partial\Omega_r = \{u \in E : \|u\| = r\},$$

$$M_1(x) := \min_{w \in [\eta x, x]} \int_0^1 G(1,s) f(s,w) \nabla s, \quad x > 0,$$

$$M_2(x) := \min_{w \in [\eta x, x]} \int_0^1 G(0,s) f(s,w) \nabla s, \quad x > 0,$$

$$M_3(x) := \max_{w \in [\eta x, x]} \int_0^1 G(0,s) f(s,w) \nabla s, \quad x > 0,$$

$$M_4(x) := \max_{w \in [\eta x, x]} \int_0^1 G(1,s) f(s,w) \nabla s, \quad x > 0.$$

For eigenvalue problem (1.1)-(1.2), we have the following existence theorems for positive solutions.

Theorem 3.1. *Suppose that (H1)–(H3) hold. If $f^0_{M_4} := \lim_{x \to 0^+} M_4(x)/x$ and $f^\infty_{M_1} := \lim_{x \to \infty} M_1(x)/x$ exist and $0 < f^0_{M_4} < f^\infty_{M_1}$, then problem (1.1)-(1.2) has at least one positive solution $\frac{1}{f^\infty_{M_1}} < \lambda < \frac{1}{f^0_{M_4}}$.*

Proof. Since $\lambda < 1/f^0_{M_4}$, we know that there exists $0 < \tau$ where, for $\tau \geq x > 0$,

$$\int_0^1 G(1,s)f(s,w)\nabla s < \frac{x}{\lambda}, \quad \forall \eta x \leq w \leq x.$$

Therefore,

$$\int_0^1 G(1,s)f(s,w)\nabla s < \frac{\tau}{\lambda}, \quad \forall \eta \tau \leq w \leq \tau.$$

We show that $Au = \mu u$, $u \in P \cap \partial\Omega_\tau \Rightarrow \mu < 1$. From $u \in P \cap \partial\Omega_\tau$, we obtain $\|u\| = \tau$ and $\eta\tau \leq u(t) \leq \tau$ for all $t \in [0,1]_{\mathbb{T}}$. Thus,

$$\begin{aligned}
\mu\|u\| = \|Au\| = (Au)(1) &= \lambda \int_0^1 G(1,s)f^*(s,u(s))\nabla s \\
&= \lambda \int_0^1 G(1,s)f(s,u(s))\nabla s \\
&< \lambda \frac{\tau}{\lambda} = \tau = \|u\|.
\end{aligned}$$

Therefore $\mu < 1$. By Theorem 2.4 (A1) it follows that

$$i(A, P \cap \Omega_\tau, P) = 1. \tag{3.1}$$

Further, since $\frac{1}{f^\infty_{M_1}} < \lambda$, there exists $0 < \tau < \rho$ where, for $\rho \leq x$,

$$\int_0^1 G(1,s)f(s,w)\nabla s > \frac{x}{\lambda}, \quad \forall \eta x \leq w \leq x.$$

Therefore,

$$\int_0^1 G(1,s)f(s,w)\nabla s > \frac{\rho}{\lambda}, \quad \forall \eta \rho \leq w \leq \rho. \tag{3.2}$$

Taking $y \equiv \rho$, we show that $x - Ax \neq vy$ for all $x \in P \cap \partial\Omega_\rho$, $0 \leq v$. Assume to the contrary that there exist $u_0 \in P \cap \partial\Omega_\rho$, $0 \leq v_0$ such that $u_0 - Au_0 = v_0\rho$. From (2.9)-(2.12), (3.2), $\|u_0\| = \rho$ and $0 < \eta\tau < \eta\rho \leq u_0(t) \leq \rho$ for all $t \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
u_0(1) &= (Au_0)(1) + v_0\rho \\
&= \lambda \int_0^1 G(1,s)f^*(s,u_0(s))\nabla s + v_0\rho \\
&= \lambda \int_0^1 G(1,s)f(s,u_0(s))\nabla s + v_0\rho \\
&> \rho + v_0\rho = (1+v_0)\rho = (1+v_0)\|u_0\| \\
&\geq \|u_0\|,
\end{aligned}$$

which is a contradiction.

By Theorem 2.4 (A2) we have

$$i(A, P \cap \Omega_\rho, P) = 0. \tag{3.3}$$

In view of (3.1), (3.3) with the fact that $\overline{\Omega}_\tau \subset \Omega_\rho$, we obtain

$$i(A, P \cap (\Omega_\rho \backslash \overline{\Omega}_\tau), P) = i(A, P \cap \Omega_\rho, P) - i(A, P \cap \Omega_\tau, P) = 0 - 1 = -1. \tag{3.4}$$

By (3.4) and the fixed point index theory the operator A has a fixed point $u \in P \cap (\Omega_\rho \backslash \overline{\Omega}_\tau)$ with $\rho \geq \|u\| \geq \tau > 0$, therefore $0 < \eta\tau \leq \eta\|u\| \leq u(t)$ for all

$t \in [0,1]_{\mathbb{T}}$. This shows that the fixed point u is a positive solution of (1.1)-(1.2). The proof is complete. $\square$

Corollary 3.2. *Suppose that* (H1)–(H3) *hold. If* $f_{M_4}^0 := \lim_{x\to 0^+} M_4(x)/x$ *and* $f_{M_2}^\infty := \lim_{x\to\infty} M_2(x)/x$ *exist and* $0 < f_{M_4}^0 < f_{M_2}^\infty$, *then problem* (1.1)-(1.2) *has at least one positive solution* $\frac{1}{f_{M_2}^\infty} < \lambda < \frac{1}{f_{M_4}^0}$.

Theorem 3.3. *Suppose that* (H1)–(H3) *hold. If* $f_{M_4}^\infty := \lim_{x\to\infty} M_4(x)/x$ *and* $f_{M_1}^0 := \lim_{x\to 0^+} M_1(x)/x$ *exist and* $0 < f_{M_4}^\infty < f_{M_1}^0$, *then problem* (1.1)-(1.2) *has at least one positive solution* $\frac{1}{f_{M_1}^0} < \lambda < \frac{1}{f_{M_4}^\infty}$.

Proof. Since $\lambda > 1/f_{M_1}^0$, we know that there exists $0 < \tau$ where, for $\tau \ge x > 0$,

$$\int_0^1 G(1,s)f(s,w)\nabla s > \frac{x}{\lambda}, \quad \forall \eta x \le w \le x.$$

Therefore,

$$\int_0^1 G(1,s)f(s,w)\nabla s > \frac{\tau}{\lambda}, \quad \forall \eta \tau \le w \le \tau.$$

Since $\lambda < \frac{1}{f_{M_4}^\infty}$, there exists $0 < \tau < \rho$ where, for $\rho \le x$,

$$\int_0^1 G(1,s)f(s,w)\nabla s < \frac{x}{\lambda}, \quad \forall \eta x \le w \le x.$$

Therefore,

$$\int_0^1 G(1,s)f(s,w)\nabla s < \frac{\rho}{\lambda}, \quad \forall \eta \rho \le w \le \rho.$$

In the same way as in the proof of Theorem 3.1, we have

$$i(A, P \cap (\Omega_\rho \backslash \bar{\Omega}_\tau), P) = i(A, P \cap \Omega_\rho, P) - i(A, P \cap \Omega_\tau, P) = 1 - 0 = 1. \tag{3.5}$$

By (3.5) and the fixed point index theory, the operator A has a fixed point $u \in P \cap (\Omega_\rho \backslash \overline{\Omega}_\tau)$ with $\rho \ge \|u\| \ge \tau > 0$, therefore $0 < \eta\tau \le \eta\|u\| \le u(t)$ for all $t \in [0,1]_{\mathbb{T}}$. This shows that the fixed point u is a positive solution of problem (1.1)-(1.2). The proof is complete. $\square$

Corollary 3.4. *Suppose that* (H1)–(H3) *hold. If* $f_{M_4}^\infty := \lim_{x\to\infty} M_4(x)/x$ *and* $f_{M_2}^0 := \lim_{x\to 0^+} M_2(x)/x$ *exist and* $f_{M_2}^0 > f_{M_4}^\infty > 0$, *then problem* (1.1)-(1.2) *has at least one positive solution provided* $\frac{1}{f_{M_2}^0} < \lambda < \frac{1}{f_{M_4}^\infty}$.

4. Eigenvalue problem (1.1)-(1.2) for $\lambda = 1$

Theorem 4.1. *Suppose that* (H1)–(H3) *hold. If* $f(t,w)$ *satisfies* $f_{M_4}^\infty < 1 < f_{M_1}^0$, *then boundary value problem* (1.1)-(1.2) *has at least one positive solution.*

The above theorem is a special case of Theorem 3.3 when $\lambda = 1$.

Corollary 4.2. *Suppose that* (H1)–(H3) *hold. If* $f(t,w)$ *satisfies* $f_{M_4}^\infty < 1 < f_{M_2}^0$, *then the boundary value problem* (1.1)-(1.2) *has at least one positive solution.*

Theorem 4.3. *Suppose that (H1)–(H3) hold. Assume that* $f(t,w)$ *satisfies*

(H4) $\frac{1}{\eta \int_0^1 G(1,s)\nabla s} < \lim_{w\to 0^+} \frac{f(t,w)}{w} \le \infty$, *uniformly for* $t \in (0,1]_{\mathbb{T}}$.

Then boundary value problem (1.1)-(1.2) *has at least one positive solution.*

Proof. By a method similar to that used to prove Theorem 3.3 with some alterations, we can complete this proof. $\square$

Corollary 4.4. *Suppose that* (H1)–(H3) *hold. Assume that* $f(t,w)$ *satisfies*

(H5) $\frac{1}{\eta\int_0^1 G(0,s)\nabla s} < \lim_{w\to 0^+}\frac{f(t,w)}{w} \le \infty$, *uniformly for* $t \in (0,1]_{\mathbb{T}}$.

Then boundary value problem (1.1)-(1.2) *has at least one positive solution.*

5. Examples

In this section, we illustrate our results with some examples.

Example 5.1. Let $\mathbb{T} = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of nonnegative integers. Take $p(t) \equiv 1$, $\alpha = 1$, $\gamma = -1/2$, $\delta = 3$, $\beta = 1/4$, $a_1 = 1/2$, $a_2 = 1/4$, $b_1 = 1/3$, $b_2 = 1/6$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, and choose

$$f(t,w) = \frac{1}{t}\Big(\frac{w}{10} + \frac{1}{w}\Big), \quad w > 0.$$

We can see that $f(t,w)$ is singular at $t = 0$ and $w = 0$. Consider the boundary-value problem

$$u^{\Delta\nabla}(t) + \frac{1}{t}\Big(\frac{u(t)}{10} + \frac{1}{u(t)}\Big) = 0, \quad t \in \mathbb{T}, \tag{5.1}$$

$$\begin{gathered} u(0) - \frac{1}{4}u^{\Delta}(0) = \frac{1}{2}u\Big(\frac{1}{4}\Big) + \frac{1}{4}u\Big(\frac{1}{2}\Big), \\ -\frac{1}{2}u(1) + 3u^{\Delta}(1) = \frac{1}{3}u\Big(\frac{1}{4}\Big) + \frac{1}{6}u\Big(\frac{1}{2}\Big). \end{gathered} \tag{5.2}$$

It is easy to see by calculation that

$$\sum_{i=1}^{2} a_i = \frac{3}{4}, \quad \sum_{i=1}^{2} b_i = \frac{1}{2}, \quad Q(t) = t, \quad \frac{\beta + \sum_{i=1}^{m-2} a_i Q(\xi_i)}{\alpha - \sum_{i=1}^{m-2} a_i} = 2, \quad d = \frac{1}{3},$$

therefore conditions (H1),(H2) and (H3) hold. By calculations we obtain

$$\begin{aligned} G(0,s) = 6\Big[&2\min\Big\{\frac{1}{4}, s\Big\} + \min\Big\{\frac{1}{2}, s\Big\} + 1 + \frac{1}{3}\min\Big\{\frac{1}{4}, s\Big\} \\ &+ \frac{1}{6}\min\Big\{\frac{1}{2}, s\Big\} + \frac{1}{2}s\Big] + 4\Big[\frac{1}{2}\min\Big\{\frac{1}{4}, s\Big\} + \frac{1}{4}\min\Big\{\frac{1}{2}, s\Big\} + \frac{1}{4}\Big], \end{aligned}$$

$$G(0,s) = \begin{cases} 27s + 7, & \text{for } 0 \le s \le \frac{1}{4}, \\ \frac{33}{2}, & \text{for } s = \frac{1}{2}, \\ 18, & \text{for } s = 1, \end{cases}$$

and

$$\begin{aligned} G(1,s) = s + 9\Big[&2\min\Big\{\frac{1}{4}, s\Big\} + \min\Big\{\frac{1}{2}, s\Big\} + 1 + \frac{1}{3}\min\Big\{\frac{1}{4}, s\Big\} \\ &+ \frac{1}{6}\min\Big\{\frac{1}{2}, s\Big\} + \frac{1}{2}s\Big] + 4\Big[\frac{1}{2}\min\Big\{\frac{1}{4}, s\Big\} + \frac{1}{4}\min\Big\{\frac{1}{2}, s\Big\} + \frac{1}{4}\Big], \end{aligned}$$

$$G(1,s) = \begin{cases} 40s + 10, & \text{for } 0 \le s \le \frac{1}{4}, \\ \frac{97}{4}, & \text{for } s = \frac{1}{2}, \\ 27, & \text{for } s = 1, \end{cases}$$

$$\eta = \inf_{s\in(0,1]}\Big\{\frac{G(0,s)}{G(1,s)}\Big\} = \frac{2}{3}.$$

Observe that $\frac{w}{10}+\frac{1}{w}$ is increasing when $w \geq \sqrt{10}$ and decreasing when $w \leq \sqrt{10}$. For $x \geq \frac{\sqrt{10}}{\eta}$, we obtain

$$\begin{aligned}
M_4(x) &= \max_{w\in[\eta x,x]} \int_0^1 G(1,s)f(s,w)\nabla s \\
&= \max_{w\in[\eta x,x]} \int_0^1 \frac{G(1,s)}{s}\Big(\frac{w}{10}+\frac{1}{w}\Big)\nabla s \\
&\leq \Big(\frac{x}{10}+\frac{1}{x}\Big)\int_0^1 \frac{G(1,s)}{s}\nabla s \\
&= \frac{365}{8}\Big(\frac{x}{10}+\frac{1}{x}\Big).
\end{aligned}$$

For $x \leq \sqrt{10}$, we obtain

$$\begin{aligned}
M_1(x) &= \min_{w\in[\eta x,x]} \int_0^1 G(1,s)f(s,w)\nabla s \\
&= \min_{w\in[\eta x,x]} \int_0^1 \frac{G(1,s)}{s}\Big(\frac{w}{10}+\frac{1}{w}\Big)\nabla s \\
&\geq \Big(\frac{x}{10}+\frac{1}{x}\Big)\int_0^1 \frac{G(1,s)}{s}\nabla s \\
&= \frac{365}{8}\Big(\frac{x}{10}+\frac{1}{x}\Big)
\end{aligned}$$

and

$$\begin{aligned}
M_2(x) &= \min_{w\in[\eta x,x]} \int_0^1 G(0,s)f(s,w)\nabla s \\
&= \min_{w\in[\eta x,x]} \int_0^1 \frac{G(0,s)}{s}\Big(\frac{w}{10}+\frac{1}{w}\Big)\nabla s \\
&\geq \Big(\frac{x}{10}+\frac{1}{x}\Big)\int_0^1 \frac{G(0,s)}{s}\nabla s \\
&= 31\Big(\frac{x}{10}+\frac{1}{x}\Big).
\end{aligned}$$

Therefore,

$$\begin{gathered}
f_{M_4}^{\infty} := \lim_{x\to\infty}\frac{M_4(x)}{x} \leq \frac{365}{8}\lim_{x\to\infty}\frac{x^2+10}{10x^2} = \frac{365}{80}; \\
f_{M_1}^{0} := \lim_{x\to 0^+}\frac{M_1(x)}{x} \geq \frac{365}{8}\lim_{x\to 0^+}\frac{x^2+10}{10x^2} = \infty; \\
f_{M_2}^{0} := \lim_{x\to 0^+}\frac{M_2(x)}{x} \geq 31\lim_{x\to 0^+}\frac{x^2+10}{10x^2} = \infty.
\end{gathered}$$

By Theorem 4.1 or Corollary 4.2, problem (5.1) and (5.2) have at least one positive solution.

Example 5.2. If we set

$$f(t,w) = t\Big(\frac{w}{10}+\frac{1}{w}\Big), \quad w>0,$$

then $f(t,w)$ is singular at $w=0$. By example 5.1 and simple calculations, we have

$$\int_0^1 G(1,s)\nabla s = 20\sum_{n=2}^{\infty}\frac{1}{4^n} + 5 + \frac{97}{16} + \frac{27}{2} = \frac{1259}{48},$$
$$\int_0^1 G(0,s)\nabla s = \frac{27}{2}\sum_{n=2}^{\infty}\frac{1}{4^n} + \frac{7}{2} + \frac{33}{8} + 9 = \frac{71}{4}.$$

Thus,

$$\frac{1}{\eta\int_0^1 G(1,s)\nabla s} = \frac{72}{1259} < \lim_{w\to 0^+}\frac{f(t,w)}{w} = t\lim_{w\to 0^+}\frac{w^2+10}{10w^2} = \infty,$$

uniformly for $t \in (0,1]_{\mathbb{T}}$;

$$\frac{1}{\eta\int_0^1 G(0,s)\nabla s} = \frac{6}{71} < \lim_{w\to 0^+}\frac{f(t,w)}{w} = \infty,$$

uniformly for $t \in (0,1]_{\mathbb{T}}$. By Theorem 4.3 or Corollary 4.4, the singular boundary value problem

$$u^{\Delta\nabla}(t) + t\Big(\frac{u(t)}{10} + \frac{1}{u(t)}\Big) = 0, \quad t \in \mathbb{T},$$
$$u(0) - \frac{1}{4}u^{\Delta}(0) = \frac{1}{2}u\big(\frac{1}{4}\big) + \frac{1}{4}u\Big(\frac{1}{2}\Big), \quad -\frac{1}{2}u(1) + 3u^{\Delta}(1) = \frac{1}{3}u\big(\frac{1}{4}\big) + \frac{1}{6}u\big(\frac{1}{2}\big)$$

has at least one positive solution.

Conclusion. In this article we have considered a singular multi-point dynamic eigenvalue problem on time scales. We have allowed $f(t,w)$ to be singular at $t=0$ and $w=0$. We have found eigenvalue intervals in which there exists at least one positive solution of problem (1.1)-(1.2). We have constructed the Green's function and have given some lemmas based on the positivity of the Green's function. Our results generalize and improve the results in [19]. Moreover, we have given two examples to indicate just how our results differ from and generalize those in other recent papers.

Acknowledgments. The author would like to thank the anonymous referees and editor for their helpful comments and suggestions.

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