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On the existence of positive solutions for the second-order boundary value problem

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ABSTRACT

This paper is concerned with the existence of positive solutions to a second order boundary value problem. By imposing growth conditions on f and using a generalization of the Leggett–Williams fixed point theorem, we prove the existence of at least three symmetric positive solutions.

1. Introduction

In the past 20 years, there has been attention focused on the existence of positive solutions to boundary value problems for ordinary differential equations; see [1–15]. It is well known that the Krasnosel'skii [16] fixed point theorems and the Leggett–Williams [17] multiple fixed-point theorem play an extremely important role.

In this paper, we discuss the existence of at least three positive solutions to the following boundary value problem:

$$u''(t) + f(u(t)) = 0, \quad t \in [0, 1], \tag{1.1}$$

$$u'(0) = 0, \qquad u(1) = 0, \tag{1.2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. A solution $u \in C^{(2)}[0, 1]$ of (1.1), (1.2) is both nonnegative and concave on [0,1]. We impose growth conditions on f which allows us to apply the generalization of the Leggett–Williams fixed point theorem in finding three symmetric positive solutions of (1.1), (1.2).

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2. Preliminaries

In this section, we give some background material concerning cone theory in a Banach space, and we then state the generalization of the Leggett–Williams fixed-point theorem.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following two conditions:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;
(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone P in a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P in a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ, β, θ be nonnegative continuous convex functionals on P, and α, ψ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers h, a, b, d and c, we define the following convex sets:

$$\begin{aligned}
&P(\gamma, c) = \{u \in P : \gamma(u) < c, \},\\
&P(\gamma, \alpha, a, c) = \{u \in P : a \leq \alpha(u),\ \gamma(u) \leq c\},\\
&Q(\gamma, \beta, d, c) = \{u \in P : \beta(u) \leq d,\ \gamma(u) \leq c\},\\
&P(\gamma, \theta, \alpha, a, b, c) = \{u \in P : a \leq \alpha(u), \theta(u) \leq b, \gamma(u) \leq c\},\\
&Q(\gamma, \beta, \psi, h, d, c) = \{u \in P : h \leq \psi(u), \beta(u) \leq d,\ \gamma(u) \leq c\}.
\end{aligned}$$

We consider the two-point boundary value problem

$$-u'' = h(t), \quad t \in [0, 1], \tag{2.1}$$

$$u'(0) = 0, \qquad u(1) = 0. \tag{2.2}$$

Lemma 2.1. *Let $h \in L^1[0, 1]$. Then the two-point boundary value problem (2.1) and (2.2) has a unique solution*

$$u(t) = \int_0^1 G(t, s)h(s)ds$$

where Green's function $G(t, s)$ is

$$G(t, s) = \begin{cases} 1 - t, & 0 \leq s \leq t \leq 1, \\ 1 - s, & 0 \leq t \leq s \leq 1. \end{cases}$$

The following is a generalization of the Leggett–Williams fixed-point theorem which will play an important role in the proof of our main results.

Theorem 2.1 ([18]). *Let P be a cone in a real Banach space E. Suppose there exist positive numbers c and M, nonnegative continuous concave functionals α and ψ on P, and nonnegative continuous convex functionals γ, β and θ on P with*

$$\alpha(u) \leq \beta(u), \qquad \|u\| \leq M\gamma(u),$$

for all $u \in \overline{P(\gamma, c)}$. Suppose that $F : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is a completely continuous operator and that there exist nonnegative numbers h, d, a, b, with $0 < d < a$ such that:

(B1) $\{u \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(u) > a\} \neq \emptyset$ *and* $\alpha(Fu) > a$ *for* $u \in P(\gamma, \theta, \alpha, a, b, c)$;
(B2) $\{u \in Q(\gamma, \beta, \psi, h, d, c) : \beta(u) < d\} \neq \emptyset$ *and* $\beta(Fu) < d$ *for* $u \in Q(\gamma, \beta, \psi, h, d, c)$;
(B3) $\alpha(Fu) > a$, *for* $u \in P(\gamma, \alpha, a, c)$ *with* $\theta(Fu) > b$;
(B4) $\beta(Fu) < d$, *for* $u \in Q(\gamma, \beta, d, c)$ *with* $\psi(Fu) < h$.

Then F has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(u_1) < d, \qquad a < \alpha(u_2) \quad \text{and} \quad d < \beta(u_3), \quad \text{with } \alpha(u_3) < a.$$

3. Main result

In this section, we impose the growth conditions on f which allow us to apply the generalization of the Leggett–Williams fixed-point theorem in establishing the existence of at least three positive solutions of (1.1) and (1.2). We will make use of various properties of Green's function $G(t, s)$ which include

$$\int_0^1 G(t,s)ds = \frac{1-t^2}{2}, \quad \text{for } 0 \leq t \leq 1,$$

$$\int_0^{1/r} G\Big(\frac{1}{2}, s\Big)ds = \frac{1}{2r}, \qquad \int_{1-(1/r)}^{1} G\Big(\frac{1}{2}, s\Big)ds = \frac{1}{2r^2}, \quad \text{for } 2 < r,$$

$$\int_{1/r}^{1/2} G\Big(\frac{1}{2}, s\Big)ds = \frac{r-2}{4r}, \qquad \int_{1/2}^{1-(1/r)} G\Big(\frac{1}{2}, s\Big)ds = \frac{r^2-4}{8r^2}, \quad \text{for } 2 < r,$$

$$\int_{t_1}^{t_2} G(t_1, s)ds + \int_{1-t_2}^{1-t_1} G(t_1, s)ds = t_2 - t_1, \quad \text{for } 0 < t_1 < t_2 \leq \frac{1}{2},$$

$$\min_{r\in[0,1]} \frac{G(t_1, r)}{G(t_2, r)} = 1, \quad \text{for } 0 < t_1 < t_2 \leq \frac{1}{2}, \qquad \max_{r\in[0,1]} \frac{G(1/2, r)}{G(t, r)} = 1, \quad \text{for } 0 < t \leq \frac{1}{2}.$$

Let $E = C[0, 1]$ be endowed with the maximum norm, $\|u\| = \max_{t\in[0,1]} |u(t)|$. Then for $0 < t_3 \leq 1/2$, we define the cone $P \subset E$ by

$$P = \{u \in E : u \text{ is concave, symmetric, nonnegative valued on } [0,1], \min_{t\in[t_3,1-t_3]} u(t) \geq 2t_3\|u\|\}.$$

We define the nonnegative, continuous concave functionals α, ψ and nonnegative continuous convex functionals β, θ, γ on the cone P by

$$\alpha(u) = \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u(t) = u(t_1),$$

$$\beta(u) = \max_{t\in[1/r,(r-1/r)]} u(t) = u\Big(\frac{1}{2}\Big),$$

$$\gamma(u) = \max_{t\in[0,t_3]\cup[1-t_3,1]} u(t) = u(t_3),$$

$$\theta(u) = \max_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u(t) = u(t_2),$$

$$\psi(u) = \min_{t\in[1/r,(r-1/r)]} u(t) = u\Big(\frac{1}{r}\Big),$$

where t_1, t_2, and r are nonnegative numbers such that

$$0 < t_1 < t_2 \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{r} \leq t_2.$$

We see that, for all $u \in P$,

$$\alpha(u) = u(t_1) \leq u\Big(\frac{1}{2}\Big) = \beta(u), \tag{3.1}$$

$$\|u\| = u\Big(\frac{1}{2}\Big) \leq \frac{1}{2t_3} u(t_3) = \frac{1}{2t_3}\gamma(u), \tag{3.2}$$

and also that $u \in P$ is a solution of (1.1), (1.2) if and only if

$$u(t) = \int_0^1 G(t,s) f(u(s)) ds, \quad \text{for } t \in [0,1].$$

We now present our result of the paper.

Theorem 3.1. *Suppose that there exist nonnegative numbers a, b, and c such that $0 < a < b \leq \frac{ct_1}{t_2}$, and suppose that f satisfies the following growth conditions:*

(C1) $f(w) < (4r^2/(r^2-4))(a - (2c/(r(1-t_3^2))))$, *for* $(2a/r) \leq w \leq a$;
(C2) $f(w) \geq b/(t_2 - t_1)$, *for* $b \leq w \leq (t_2 b)/t_1$;
(C3) $f(w) \leq (2c)/(1-t_3^2)$, *for* $0 \leq w \leq c/(2t_3)$.

Then the boundary value problem (1.1) *and* (1.2) *has three symmetric positive solutions u_1, u_2 and u_3 satisfying*

$$\max_{t\in[0,t_3]\cup[1-t_3,1]} u_i(t) \leq c, \quad \textit{for } i = 1,2,3,$$

$$\min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u_1(t) > b, \qquad \max_{t\in[1/r,(r-1)/r]} u_2(t) < a,$$

$$\min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u_3(t) < b, \quad \textit{with} \quad \max_{t\in[1/r,(r-1)/r]} u_3(t) > a.$$

Proof. Let us define the completely continuous operator F by

$$(Fu)(t) = \int_0^1 G(t,s) f(u(s)) ds.$$

We will seek fixed points of F in the cone. We note that, if $u \in P$, then from properties of $G(t,s), Fu(t) \geq 0$, and $(Fu)''(t) = -f(u(t)) \leq 0$, $0 \leq t \leq 1$, $Fu(t_3) \geq 2t_3 Fu(1/2)$, and $Fu(t) = Fu(1-t)$, $0 \leq t \leq 1/2$. This implies that $Fu \in P$, and so $F : P \to P$.

Now, for all $u \in P$, from (3.1), we get $\alpha(u) \leq \beta(u)$ and from (3.2), $\|u\| \leq \frac{1}{2t_3}\gamma(u)$.

If $u \in \overline{P(\gamma, c)}$, then $\|u\| \leq 1/(2t_3)\gamma(u) \leq c/(2t_3)$ and from (C3) we get,

$$\begin{aligned}\gamma(Fu) &= \max_{t\in[0,t_3]\cup[1-t_3,1]} \int_0^1 G(t,s) f(u(s)) ds \\ &= \int_0^1 G(t_3,s) f(u(s)) ds \\ &\leq \Big(\frac{2c}{1-t_3^2}\Big) \int_0^1 G(t_3,s) ds = c.\end{aligned}$$

Thus, $F : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$. It is immediate that

$$\Big\{u \in P\Big(\gamma, \theta, \alpha, b, \frac{bt_2}{t_1}, c\Big) : \alpha(u) > b\Big\} \neq \emptyset \quad \text{and} \quad \Big\{u \in Q\Big(\gamma, \beta, \psi, \frac{2a}{r}, a, c\Big) : \beta(u) < a\Big\} \neq \emptyset.$$

We will show the remaining conditions of Theorem 2.1.

(1) If $u \in Q(\gamma, \beta, a, c)$ with $\psi(Fu) < (2a)/r$ then $\beta(Fu) < a$.

$$\begin{aligned}
\beta(Fu) &= \max_{t \in [1/r,(r-1)/r]} \int_0^1 G(t, s) f(u(s)) ds \\
&= \int_0^1 G\Big(\frac{1}{2}, s\Big) f(u(s)) ds \\
&= \int_0^1 \frac{G(1/2, s)}{G(1/r, s)} G\Big(\frac{1}{r}, s\Big) f(u(s)) ds \\
&\leq \int_0^1 G\Big(\frac{1}{r}, s\Big) f(u(s)) ds = \psi(Fu) < a.
\end{aligned}$$

(2) If $u \in Q(\gamma, \beta, \psi, (2a)/r, a, c)$, then $\beta(Fu) < a$.

$$\begin{aligned}
\beta(Fu) &= \max_{t \in [1/r,(r-1)/r]} \int_0^1 G(t, s) f(u(s)) ds \\
&= \int_0^1 G\Big(\frac{1}{2}, s\Big) f(u(s)) ds \\
&= 2 \int_0^{1/r} G\Big(\frac{1}{2}, s\Big) f(u(s)) ds + 2 \int_{1/r}^{1/2} G\Big(\frac{1}{2}, s\Big) f(u(s)) ds \\
&< \frac{2c}{r(1 - t_3^2)} + \Big(\frac{4r^2}{r^2 - 4}\Big)\Big(a - \frac{2c}{r(1 - t_3^2)}\Big)\Big(\frac{r^2 - 4}{4r^2}\Big) = a.
\end{aligned}$$

(3) If $u \in Q(\gamma, \alpha, b, c)$ with $\theta(Fu) > (bt_2)/t_1$, then $\alpha(Fu) > b$.

$$\begin{aligned}
\alpha(Fu) &= \min_{t \in [t_1,t_2] \cup [1-t_2,1-t_1]} \int_0^1 G(t, s) f(u(s)) ds \\
&= \int_0^1 G(t_1, s) f(u(s)) ds \\
&= \int_0^1 \frac{G(t_1, s)}{G(t_2, s)} G(t_2, s) f(u(s)) ds \\
&\geq \int_0^1 G(t_2, s) f(u(s)) ds = \theta(Fu) > b.
\end{aligned}$$

(4) If $u \in Q(\gamma, \theta, \alpha, b, (bt_2)/t_1, c)$, then $\alpha(Fu) > b$.

$$\begin{aligned}
\alpha(Fu) &= \min_{t \in [t_1,t_2] \cup [1-t_2,1-t_1]} \int_0^1 G(t, s) f(u(s)) ds \\
&= \int_0^1 G(t_1, s) f(u(s)) ds \\
&> \int_{t_1}^{t_2} G(t_1, s) f(u(s)) ds + \int_{1-t_2}^{1-t_1} G(t_1, s) f(u(s)) ds
\end{aligned}$$

$$\geq \left(\frac{b}{t_2-t_1}\right)\int_{t_1}^{t_2} G(t_1,s)ds + \left(\frac{b}{t_2-t_1}\right)\int_{1-t_2}^{1-t_1} G(t_1,s)ds$$
$$= \left(\frac{b}{t_2-t_1}\right)\left(\frac{-2t_1+t_1^2+2t_2-t_2^2}{2}\right) + \left(\frac{b}{t_2-t_1}\right)\left(\frac{-t_1^2+t_2^2}{2}\right) = b.$$

Since all the conditions of the generalized Leggett–Williams fixed point theorem are satisfied, (1.1), (1.2) has three positive solutions $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that

$$\alpha(u_1) > b, \qquad \beta(u_2) < a, \qquad \alpha(u_3) < b, \quad \text{with } \beta(u_3) > a. \quad \square$$

Remark 3.1. When f is autonomous, we select to carry out the analysis. But, if $f = f(t, u(t))$ and moreover, for each fixed u, $f(t, u(t))$ is symmetric about $t = \frac{1}{2}$, then a similar theorem would be correct with respect to same cone P.

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