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POSITIVE SOLUTIONS FOR AN m-POINT BOUNDARY-VALUE PROBLEM

LE XUAN TRUONG, LE THI PHUONG NGOC, NGUYEN THANH LONG

Abstract. In this paper, we obtain sufficient conditions for the existence of a positive solution, and infinitely many positive solutions, of the m-point boundary-value problem

$$x''(t) = f(t, x(t)), \quad 0 < t < 1,$$

$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i)\,.$$

Our main tools are the Guo-Krasnoselskii's fixed point theorem and the monotone iterative technique. We also show that the set of positive solutions is compact.

1. Introduction

The existence and multiplicity of positive solutions for boundary-value problems have been extensively studied by many authors using various techniques, such fixed point theorem in cones, the nonlinear alternative of Leray-Schauder, the Leggett-William's fixed point theorem, monotone iterative techniques. We refer the reader to the references in this article and the references therein for the results of multi-point boundary-value problems.

Han [5] studied the existence of positive solutions for the three-point boundary-value problem at resonance

$$x''(t) = f(t, x(t)), \quad 0 < t < 1, \tag{1.1}$$

$$x'(0) = 0, \quad x(\eta) = x(1), \tag{1.2}$$

where $\eta \in (0, 1)$. The main tool is the fixed point theorem in cones. By the same method, Long and Ngoc [6] have studied the equation (1.1) together with the boundary conditions

$$x'(0) = 0, \quad x(1) = \alpha x(\eta), \tag{1.3}$$

where α and η in $(0, 1)$. The authors proved the existence of positive solutions and established the compactness of the set of positive solutions.

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Based on the above works, we investigate the m-point boundary-value problem consisting of the equation (1.1) together with the boundary conditions

$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \tag{1.4}$$

where $m \geq 3$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ and $\alpha_i \geq 0$, for all $i = 1, 2, \dots m-2$ such that $\sum_{i=1}^{m-2} \alpha_i < 1$. We shall establish the existence and multiplicity of positive solutions by applying well-know Guo-Krasnoselskii's fixed point theorem and applying the monotone iterative technique.

Let $\beta \in (0, \frac{\pi}{2})$. Obviously, problem (1.1), (1.4) is equivalent to the problem

$$x''(t) + \beta^2 x(t) = g(t, x(t)), \tag{1.5}$$

$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \tag{1.6}$$

where

$$g(t, x) = f(t, x) + \beta^2 x. \tag{1.7}$$

In this paper, we sue the following assumptions:

(H1) $\sum_{i=1}^{m-2} \alpha_i \cos \beta\eta_i - \cos \beta > 0$;

(H2) $f : [0, 1] \times [0, +\infty) \to \mathbb{R}$ is a continuous function such that

$$f(t, x) \geq -\beta^2 x, \forall t \in [0, 1], x \in [0, +\infty); \tag{1.8}$$

(H2') The function $f(t, x)$ is nondecreasing in x and satisfy (H2)

We put:

$$K_m = \frac{1}{\sum_{i=1}^{m-2} \alpha_i \cos \beta\eta_i - \cos \beta};$$

$$M = \frac{\sin \beta}{\beta}(1 + K_m);$$

$$M_0 = \frac{K_m \cos \beta}{\beta}\Big(1 - \sum_{i=1}^{m-2} \alpha_i\Big) \sin \beta(1 - \eta_{m-2}).$$

The main results for the existence and multiplicity of positive solutions are the following theorems, in which the operator T and constant c, $0 < c < 1$ will be defined in next section. Applying well-know Guo-Krasnoselskii's fixed point theorem, we obtain the following result.

Theorem 1.1. *Let* (H1)–(H2) *hold. If there exist two constants R_1, R_2 such that $R_1 < cR_2$ and one the following two conditions is satisfied:*

$$\begin{gathered} f(t, x) + \beta^2 x \leq \frac{R_1}{M}, \quad \forall (t, x) \in [0, 1] \times [cR_1, R_1], \\ f(t, x) + \beta^2 x \geq \frac{R_2}{M_0 \eta_{m-2}}, \quad \forall (t, x) \in [0, 1] \times [cR_2, R_2], \end{gathered} \tag{1.9}$$

or

$$\begin{gathered} f(t, x) + \beta^2 x \geq \frac{R_1}{M_0 \eta_{m-2}}, \quad \forall (t, x) \in [0, 1] \times [cR_1, R_1], \\ f(t, x) + \beta^2 x \leq \frac{R_2}{M}, \quad \forall (t, x) \in [0, 1] \times [cR_2, R_2]. \end{gathered} \tag{1.10}$$

Then the boundary-value problem (1.5)-(1.7) *has a positive solution.*

Using the monotone iterative technique, we have the following result.

Theorem 1.2. *Let* (H1), (H2') *hold. Suppose there exist two positive numbers* $R_1 < R_2$ *such that*

$$\sup_{t\in[0,1]} g(t, R_2) \leq \frac{R_2}{M}, \quad \inf_{t\in[0,1]} g(t, cR_1) \geq \frac{R_1}{M_0\eta_{m-2}}. \tag{1.11}$$

Then problem (1.5)-(1.7) *has positive solutions* x_1^*, x_2^*, x_1^* *and* x_2^* *may coincide with*

$$R_1 \leq \|x_1^*\| \leq R_2 \quad \text{and} \quad \lim_{n\to+\infty} T^n x_0 = x_1^*, \quad \text{where } x_0(t) = R_2, t \in [0,1],$$

and

$$R_1 \leq \|x_2^*\| \leq R_2 \quad \text{and} \quad \lim_{n\to+\infty} T^n \widehat{x}_0 = x_2^*, \quad \text{where } \widehat{x}_0(t) = R_1, t \in [0,1].$$

Clearly, in the above theorem, we not only obtain the existence as in Theorem 1.1, but also we establish a sequence which converges to a solution of problem (1.5)-(1.7).

This paper consists of five sections. In Section 2, we present the lemmas that will be used to prove the existence results. The proofs and two corollaries of Theorems 1.1, 1.2 will be given in Section 3. In Section 4, we give sufficient conditions for existence of infinitely many positive solutions, furthermore, an example is also given here. Finally, in section 5, we show that the set of positive solutions is compact.

2. Preliminaries

Consider the Banach spaces $C[0,1]$ and $C^2[0,1]$ equipped with the norms

$$\|x\| = \max\{|x(t)| : 0 \leq t \leq 1\},$$
$$\|x\|_2 = \max\{\|x\|, \|x'\|, \|x''\|\},$$

respectively. We define a linear operator $L : D(L) \subset C^2[0,1]] \to C[0,1]$ by setting

$$Lx := x'' + \beta^2 x, \tag{2.1}$$

in which $D(L) = \{x \in C^2[0,1] : x'(0) = 0,\ x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i)\}$. We shall proceed with some properties of the inverse operator of L.

Lemma 2.1. *Let* $\beta \in (0, \frac{\pi}{2})$. *Then for each* $h \in C[0,1]$, *there a unique function* $x = A(h) \in D(L)$ *such that* $Lx = h$ *in* $(0,1)$. *The function* $A(h)$ *is defined by*

$$Ah(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.2}$$

where

$$
\begin{aligned}
G(t,s) &= \begin{cases} \frac{1}{\beta}\sin\beta(t-s), & 0 \le s \le t \le 1,\\ 0, & 0 \le t \le s \le 1 \end{cases}\\
&+ \frac{K_m}{\beta}\cos\beta t \begin{cases} \sin\beta(1-s) - \sum_{i=1}^{m-2}\alpha_i \sin\beta(\eta_i - s), & 0 \le s \le \eta_1,\\ \sin\beta(1-s) - \sum_{i=2}^{m-2}\alpha_i \sin\beta(\eta_i - s), & \eta_1 \le s \le \eta_2,\\ \sin\beta(1-s) - \sum_{i=3}^{m-2}\alpha_i \sin\beta(\eta_i - s), & \eta_2 \le s \le \eta_3,\\ \dots \\ \sin\beta(1-s) - \sum_{i=k}^{m-2}\alpha_i \sin\beta(\eta_i - s), & \eta_{k-1} \le s \le \eta_k,\\ \dots \\ \sin\beta(1-s), & \eta_{m-2} \le s \le 1. \end{cases}
\end{aligned}
\tag{2.3}
$$

Lemma 2.2. *Let $\beta \in (0, \frac{\pi}{2})$. We have*

(i) *The operator $A : C[0,1] \to C[0,1]$ is completely continuous linear operator.*
(ii) *For any positive function $h \in C[0,1]$, the function Ah is also positive.*

The proof of Lemmas 2.1, 2.2 are straightforward and we will omit them. Now, we shall establish some estimations for the Green function $G(t,s)$.

Lemma 2.3. *Let $\beta \in (0, \frac{\pi}{2})$ and (H1) hold. Then*

(i) *$0 \le G(t,s) \le M$ for all $(t,s) \in [0,1] \times [0,1]$.*
(ii) *$G(t,s) \ge M_0$, for all $(t,s) \in [0,1] \times [0, \eta_{m-2}]$.*
(iii) *There exist a constant $c \in (0,1)$ and a continuous function $\Phi : [0,1] \to [0,+\infty)$ such that*

$$
c\Phi(s) \le G(t,s) \le \Phi(s), \quad \textit{for all } t,s \in [0,1].
$$

Proof. **Part (i).** From the Green function $G(t,s)$, we deduce that

$$
0 \le G(t,s) \le \frac{\sin\beta}{\beta}(1 + K_m) \equiv M, \quad \forall (t,s) \in [0,1] \times [0,1]. \tag{2.4}
$$

Part (ii). Put $\eta_0 = 0$, $\eta_{m-1} = 1$. For all $t \in [0,1]$ and $s \in [\eta_{k-1}, \eta_k]$, we have

$$
\begin{aligned}
G(t,s) &\ge \frac{K_m \cos\beta t}{\beta}\Big[\sin\beta(1-s) - \sum_{i=k}^{m-2}\alpha_i \sin\beta(\eta_i - s)\Big]\\
&\ge \frac{K_m \cos\beta t}{\beta}\Big[\sin\beta(1-s) - \sum_{i=k}^{m-2}\alpha_i \sin\beta(1 - s)\Big]\\
&\ge \frac{K_m \cos\beta t}{\beta}\Big(1 - \sum_{i=k}^{m-2}\alpha_i\Big)\sin\beta(1-s)\\
&\ge \frac{K_m \cos\beta}{\beta}\Big(1 - \sum_{i=1}^{m-2}\alpha_i\Big)\sin\beta(1-\eta_{m-2}) \equiv M_0.
\end{aligned}
\tag{2.5}
$$

Since the above inequality holds for $k = 1, 2, \dots, m-2$, the proof part (ii) is complete.

Part (iii). Let

$$
H(t,s) = \mu(1-s) - G(t,s).
$$

We shall show that when $\mu > 0$ sufficiently large,

$$H(t,s) \ge 0, \quad \forall (t,s) \in [0,1] \times [0,1], \tag{2.6}$$

and that when $\mu > 0$ sufficiently small,

$$H(t,s) \le 0, \quad \forall (t,s) \in [0,1] \times [0,1]. \tag{2.7}$$

To prove (2.6), we use that from (2.3), for all $t, s \in [0,1]$,

$$G(t,s) \le \frac{1}{\beta} \sin\beta(1-s) + \frac{K_m}{\beta} \sin\beta(1-s) \le (K_m + 1)(1-s); \tag{2.8}$$

therefore

$$H(t,s) \ge \mu(1-s) - (K_m + 1)(1-s) = (\mu - K_m - 1)(1-s). \tag{2.9}$$

So, if we choose $\mu \equiv \mu_1 \ge K_m + 1$ then $H(t,s) \ge 0$, for all $t, s \in [0,1]$.

To prove of (2.7), we consider two cases:

Case 1: $s \in [0, \eta_{m-2}]$. From (2.5) we can deduce that, for all $t \in [0,1]$,

$$H(t,s) = \mu(1-s) - G(t,s) \le \mu(1-s) - M_0 \le \mu - M_0. \tag{2.10}$$

So, for $\mu \le M_0$, we have $H(t,s) \le 0$, for all $t \in [0,1]$, $s \in [0, \eta_{m-2}]$.

Case 2: $s \in [\eta_{m-2}, 1]$. The function $z \mapsto \frac{\sin z}{z}$ is decreasing on $(0, \pi]$, so we obtain

$$\frac{\sin\beta(1-s)}{\beta(1-s)} \ge \frac{\sin\beta(1-\eta_{m-2})}{\beta(1-\eta_{m-2})}.$$

Therefore,

$$\begin{aligned} H(t,s) &= \mu(1-s) - G(t,s) \\ &\le \mu(1-s) - \frac{K_m \cos\beta}{\beta} \sin\beta(1-s) \\ &\le \mu(1-s) - K_m \cos\beta \frac{\sin\beta(1-s)}{\beta(1-s)}(1-s) \\ &\le \Big[\mu - K_m \cos\beta \frac{\sin\beta(1-\eta_{m-2})}{\beta(1-\eta_{m-2})}\Big](1-s). \end{aligned} \tag{2.11}$$

If we choose $\mu \le K_m \cos\beta \frac{\sin\beta(1-\eta_{m-2})}{\beta(1-\eta_{m-2})} \equiv M_1$, then $H(t,s) \le 0$, for all $t \in [0,1]$, $s \in [\eta_{m-2}, 1]$.

Hence, for $\mu \equiv \mu_2 \le \min\{M_0, M_1\} = M_0$, we have $H(t,s) \le 0$, for all $t, s \in [0,1]$. Finally, by letting $\Phi(s) = \mu_1(1-s)$ and $c = \frac{\mu_2}{\mu_1}$, the part *(iii)* of this lemma is proved. □

Let K be the cone in $C[0,1]$, consisting of all nonnegative functions and

$$P = \{x \in K : x(t) \ge c\|x\|, \ \forall t \in [0,1]\}.$$

It is clear that P is also a cone in $C[0,1]$. For each $x \in P$, denote $F(x)(t) = g(t, x(t))$, $t \in [0,1]$. From the assumption (H2) we deduce that the operator $F : P \to K$ is continuous. Therefore, the operator $T \equiv A \circ F : P \to K$ is a completely continuous. On the other hand, for $x \in P$, by Lemma 2.3 we have

$$Tx(t) = \int_0^1 G(t,s)F(x)(s)ds \ge c\int_0^1 \Phi(s)F(x)(s)ds, \tag{2.12}$$

$$\|Tx\| = \max_{0 \le t \le 1} \int_0^1 G(t,s)F(x)(s)ds \le \int \Phi(s)F(x)(s)ds, \tag{2.13}$$

which implies

$$Tx(t) \geq c\|Tx\|. \tag{2.14}$$

Hence, we have the following result.

Lemma 2.4. *The operator $T \equiv A \circ F : P \to P$ is a completely continuous operator.*

It is easy to verify the nonzero fixed points of the operator T are the positive solutions of the problem (1.5)-(1.7).

3. Proofs and corollaries of main Theorems

At first, by using the same method as in [5] and the monotone iterative technique, combining with Lemmas 2.1–2.4, we prove Theorems 1.1 and 1.2. For the convenience of the reader, let us state the following Guo-Krasnoselskii's fixed point theorem [3].

Theorem 3.1 (Guo-Krasnoselskii)**.** *Let X be a Banach space, and let $P \subset X$ be a cone. Assume Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that*

(i) *$\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, or*
(ii) *$\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Proof of Theorem 1.1. Let

$$\Omega_1 = \{x \in C[0,1]| : \|x\| < R_1\}, \quad \Omega_2 = \{x \in C[0,1]| : \|x\| < R_2\}.$$

Then Ω_1, Ω_2 are open bounded subsets of $C[0,1]$ with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$.

Case (1.9)**.** For $x \in P$ with $\|x\| = R_1$, we have

$$g(s, x(s)) = f(s, x(s)) + \beta^2 x(s) \leq \frac{R_1}{M} = \frac{\|x\|}{M}.$$

So

$$\|Tx\| = \max_{t \in [0,1]} \int_0^1 G(t,s) g(s, x(s)) ds \leq \frac{\|x\|}{M} ax_{t\in[0,1]} \int_0^1 G(t,s) ds \leq \|x\|.$$

This implies

$$\|Tx\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_1. \tag{3.1}$$

On the other hand, for $x \in P$ with $\|x\| = R_2$, we have

$$\begin{aligned} Tx(t) &= \int_0^1 G(t,s) \Big(f(s, x(s)) + \beta^2 x(s) \Big) ds \\ &\geq \frac{R_2}{M_0 \eta_{m-2}} \int_0^{\eta_{m-2}} G(t,s) ds \geq R_2 = \|x\|, \end{aligned}$$

accordingly

$$\|Tx\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2. \tag{3.2}$$

By (3.1)-(3.2) and the first part of Theorem 3.1, it follows that T has a fixed point x_0 in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, such x_0 is a positive solution of (1.5)-(1.7).

Case (1.10)**.** In this case, using the same method, by the second part of Theorem 3.1, we obtain the same result as above.

The proof is complete. □

Proof of Theorem 1.2. We define

$$P_{[R_1,R_2]} = \{x \in P : R_1 \le \|x\| \le R_2\}.$$

Let $x \in P_{[R_1,R_2]}$, then $cR_1 \le c\|x\| \le x(t) \le \|x\| \le R_2$, for all $t \in [0,1]$. So, we have

$$Tx(t) = \int_0^1 G(t,s)g(s,x(s))ds \le \frac{R_2}{M}\int_0^1 G(t,s)ds \le R_2,$$
$$Tx(t) = \int_0^1 G(t,s)g(s,x(s))ds \ge \int_0^1 G(t,s)g(s,cR_1)ds \ge R_1,$$

which implies $TP_{[R_1,R_2]} \subset P_{[R_1,R_2]}$.

Now, let $x_0(t) = R_2, t \in [0,1]$ then $x_0 \in P_{[R_1,R_2]}$. We put

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad n = 1,2,\dots \tag{3.3}$$

Since $TP_{[R_1,R_2]} \subset P_{[R_1,R_2]}$ we have $x_n \in P_{[R_1,R_2]}$, for all $n \in \mathbb{Z}_+$. By the Lemma 2.4, we can deduce that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to+\infty} x_{n_k} = x_1^* \in P_{[R_1,R_2]}. \tag{3.4}$$

On the other hand, from the assumption (H2'), it is clear that $T : P_{[R_1,R_2]} \to P_{[R_1,R_2]}$ is nondecreasing. Therefore, since

$$0 \le x_1(t) \le \|x_1\| \le R_2 = x_0(t), t \in [0,1],$$

we have $Tx_1 \le Tx_0, \quad i.e., x_2 \le x_1$. By induction, then

$$x_{n+1} \le x_n, \quad \text{for all } n = 1,2,\dots. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\lim_{n\to+\infty} x_n = x_1^*. \tag{3.6}$$

Letting $n \to +\infty$ in (3.3) yields $Tx_1^* = x_1^*$.

Let $\widehat{x}_0(t) = R_1, t \in [0,1]$ and $\widehat{x}_{n+1} = T\widehat{x}_n$, $n = 1,2,\dots$. We have $\widehat{x}_0 \in P_{[R_1,R_2]}$ which implies that $\widehat{x}_n \in P_{[R_1,R_2]}$, for all $n \in \mathbb{Z}_+$. Moreover, from the assumptions of Theorem 1.2 and from the definition of the operator T,

$$\widehat{x}_1(t) = T\widehat{x}_0(t) \ge \int_0^1 G(t,s)g(s,cR_1)ds \ge R_1 = \widehat{x}_0(t), \quad t \in [0,1].$$

Therefore, by using the arguments as above, we deduce that $\widehat{x}_n \to x_2^* \in P_{[R_1,R_2]}$ and $Tx_2^* = x_2^*$. The proof is complete. $\square$

Next, in order to present the first corollary, we will use the following notation:

$$f^0 = \limsup_{x\to 0^+} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \quad f^\infty = \limsup_{x\to+\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$f_0 = \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \quad f_\infty = \liminf_{x\to+\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x}.$$

Corollary 3.2. *Let* (H1)–(H2) *hold. Then the boundary-value problem* (1.5)-(1.7) *has at least one positive solution in the case*

(i) $f^0 \le -\beta^2 + \frac{1}{M}$ *and* $f_\infty \ge \frac{1}{M_0\eta_{m-2}}$, *(in particular* $f^0 = -\beta^2$, $f_\infty = \infty$*); or*

(ii) $f_0 \ge \frac{1}{M_0\eta_{m-2}}$ *and* $f_\infty \le -\beta^2 + \frac{1}{M}$, *(in particular* $f_0 = \infty$, $f^\infty = -\beta^2$*).*

Proof. It is easy to verify that conditions of Theorem 1.1 can be obtained from conditions (i) or (ii) of this corollary. We omit the proof. $\square$

We close this section with the following result.

Corollary 3.3. *Let* (H1), (H2') *hold. Further assume*

$$\liminf_{x\to+\infty}\sup_{t\in[0,1]}\frac{f(t,x)}{x}\le-\beta^2+\frac{1}{M},\quad\Big(\textit{in particular }\liminf_{x\to+\infty}\sup_{t\in[0,1]}\frac{f(t,x)}{x}=-\beta^2\Big) \tag{3.7}$$

and

$$\limsup_{x\to0^+}\inf_{t\in[0,1]}\frac{f(t,x)}{x}\ge\frac{1}{M_0\eta_{m-2}},\quad\Big(\textit{in particular }\limsup_{x\to0^+}\inf_{t\in[0,1]}\frac{f(t,x)}{x}=+\infty\Big). \tag{3.8}$$

Then there exist two constants $0<R_1<R_2$ *such that the problem* (1.5)-(1.7) *has positive solutions* x_1^*, x_2^* *(*x_1^* *and* x_2^* *may coincide) with*

$$R_1\le\|x_1^*\|\le R_2\quad\textit{and}\quad\lim_{n\to+\infty}T^nx_0=x_1^*,\quad\textit{where } x_0(t)=R_2,\ t\in[0,1], \tag{3.9}$$

$$R_1\le\|x_2^*\|\le R_2\quad\textit{and}\quad\lim_{n\to+\infty}T^n\widehat{x}_0=x_2^*,\quad\textit{where } \widehat{x}_0(t)=R_1,\ t\in[0,1]. \tag{3.10}$$

Clearly, from the assumptions of this corollary, the conditions of Theorem 1.2 hold. So we omit the proof.

4. Existence of infinitely many positive solutions

In this section we give sufficient conditions for existence of infinitely many positive solutions. For this purpose, we assume that there exists a sequence $\{R_n\}_{n=1}^{\infty}\subset\mathbb{R}$ such that $0<R_n<cR_{n+1}$ and for all $n\in\mathbb{N}$,

(H3) $f(t,x)+\beta^2x\le\frac{R_{2n+1}}{M}$, for all $(t,x)\in[0,1]\times[cR_{2n-1},R_{2n-1}]$,
(H4) $f(t,x)+\beta^2x\ge\frac{R_{2n}}{M_0\eta_{m-2}}$, for all $(t,x)\in[0,1]\times[cR_{2n},R_{2n}]$.

Theorem 4.1. *Assume* (H1)–(H4) *hold. Then the boundary-value problem* (1.5)-(1.7) *has infinitely many positive solutions* $\{x_n\}_{n\in\mathbb{N}}$ *satisfying* $R_{2n-1}\le\|x_n\|\le R_{2n}$*, for* $n\in\mathbb{N}$.

Proof. Let $\Omega_n=\{x\in C[0,1]:\|x\|<R_n\}$. Then $0\in\Omega_n$ and $\overline{\Omega_n}\subset\Omega_{n+1}$, for $n\in\mathbb{N}$. In the following, we show that for all $n\in\mathbb{N}$,

$$\|Tx\|\le\|x\|,\quad\forall x\in P\cap\partial\Omega_{2n-1}, \tag{4.1}$$

$$\|Tx\|\ge\|x\|,\quad\forall x\in P\cap\partial\Omega_{2n}. \tag{4.2}$$

First, for $x\in P\cap\partial\Omega_{2n-1}$, $s\in[0,1]$, we have

$$cR_{2n-1}=c\|x\|\le x(s)\le\|x\|=R_{2n-1}. \tag{4.3}$$

So, by the assumption (H3),

$$g(s,x(s))\le\frac{R_{2n-1}}{M}. \tag{4.4}$$

Consequently,

$$\|Tx\|=\max_{t\in[0,1]}\int_0^1G(t,s)g(s,x(s))ds\le R_{2n-1}=\|x\|,\quad t\in[0,1], \tag{4.5}$$

which implies (4.1).

Next, for $x \in P \cap \partial\Omega_{2n}$ and $s \in [0,1]$, we have $cR_{2n} \leq x(s) \leq R_{2n}$. Then, from (H4) it follows that for $t \in [0,1]$,

$$Tx(t) = \int_0^1 G(t,s)g(s,x(s))ds \geq \int_0^{\eta_{m-2}} G(t,s)g(s,x(s))ds \geq R_{2n} = \|x\|. \quad (4.6)$$

So, we obtain (4.2). The inequalities (4.1) and (4.2) prove that T satisfies condition (i) of Therorem 1.1 in $P \cap (\overline{\Omega}_{2n} \backslash \Omega_{2n-1})$. Therefore T has a fixed point $x_n \in P \cap (\overline{\Omega}_{2n} \backslash \Omega_{2n-1})$. This implies that $R_{2n-1} \leq \|x_n\| \leq R_{2n}$. This completes the proof. □

Example. Let $\alpha, \beta \in \mathbb{R}_+$ and $\rho : \mathbb{R}_+ \to \mathbb{R}$ such that $\alpha x \leq \rho(x) \leq \beta x$, for all $x \in \mathbb{R}_+$. We consider the function $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$, defined by

$$f(t,x) = \begin{cases} f_1(t,x) & \text{if } R_{2n-2} \leq x \leq cR_{2n-1}, \\ f_2(t,x) & \text{if } cR_{2n-1} \leq x \leq R_{2n-1}, \\ f_3(t,x) & \text{if } R_{2n-1} \leq x \leq cR_{2n}, \\ f_4(t,x) & \text{if } cR_{2n} \leq x \leq R_{2n}, \end{cases}$$

for all $n \in \mathbb{N}$, where $R_0 = 0$, $\{R_n\}_{n=1}^{+\infty} \subset \mathbb{R}$ such that $0 < R_n < cR_{n+1}$ and

$$\begin{gathered} f_1(t,x) = \frac{cR_{2n-1} - x}{cR_{2n-1} - R_{2n-2}} f_4(t,x) + \frac{R_{2n-2} - x}{R_{2n-2} - cR_{2n-1}} f_2(t,x), \\ f_2(t,x) = t\rho\Big(\frac{x}{\beta M}\Big) - \beta^2 x, \\ f_3(t,x) = \frac{cR_{2n} - x}{cR_{2n} - R_{2n-1}} f_2(t,x) + \frac{R_{2n-1} - x}{R_{2n-1} - cR_{2n}} f_4(t,x), \\ f_4(t,x) = (t+1)\rho\Big(\frac{x}{\alpha M_0 \eta_{m-2}}\Big) - \beta^2 \ln(1+x). \end{gathered}$$

It is clear that f is a function continuous on $[0,1] \times [0,+\infty)$. Since the inequality $x - \ln(1+x) \geq 0$, $\forall x \geq 0$, so $f(t,x) + \beta^2 x \geq 0$, $\forall t \in [0,1], x \in [0,+\infty)$. Moreover, by the properties of the function ρ, we can deduce that the assumptions (H3) and (H4) of Theorem 4.1 hold.

5. Compactness of the set of positive solutions

Theorem 5.1. *Let* (H1)–(H2) *hold. In addition, suppose that there exists a constant* $\alpha \in (0,1)$ *such that*

$$f_0 \geq \frac{1}{M\eta_{m-2}} \quad \text{and} \quad f^\infty \leq -\beta^2 + \frac{\alpha}{M} \quad \Big(\text{in particular } f_0 = \infty,\ f^\infty = -\beta^2\Big). \quad (5.1)$$

Then the set of positive solutions of the problem (1.5)–(1.7) *is nonempty and compact.*

Proof. Put $\Sigma = \{x \in P : x = Tx\}$. By Theorem 1.1, we have $\Sigma \neq \emptyset$. We shall show that Σ is compact. From assumption (5.1), there exists $R > 0$ such that

$$f(t,x) \leq \Big(-\beta^2 + \frac{\alpha}{M}\Big) x, \quad \forall t \in [0,1], x \geq R. \quad (5.2)$$

Therefore,

$$g(t,x(t)) = f(t,x(t)) + \beta^2 x(t) \leq \frac{\alpha}{M} x(t) + \gamma, \quad \forall t \in [0,1], \quad (5.3)$$

where $\gamma = \max\{g(t,x) : (t,x) \in [0,1] \times [0,R]\}$. For $x \in \Sigma$ and $t \in [0,1]$, we have

$$x(t) = \int_0^1 G(t,s) g\left(s, x(s)\right) ds \leq M \int_0^1 \Big(\frac{\alpha}{M} x(s) + \gamma\Big) ds \leq \alpha \|x\| + M\gamma, \tag{5.4}$$

so

$$\|x\| \leq \frac{M\gamma}{1-\alpha}, \quad \forall x \in \Sigma. \tag{5.5}$$

From the compactness of the operator $T : P \to P$, it follows from (5.5) that $T(\Sigma)$, and then $\Sigma \subset T(\Sigma)$ are relatively compact.

To prove Σ is closed, let $\{x_n\} \subset \Sigma$ be a sequence and $\lim_{n\to+\infty} \|x_n - \widehat{x}\| = 0$. For $t \in [0,1]$, we have

$$\begin{aligned}
&\big|\widehat{x}(t) - \int_0^1 G(t,s) g\left(s, \widehat{x}(s)\right) ds\big| \\
&\leq |\widehat{x}(t) - x_n(t)| + \big|x_n(t) - \int_0^1 G(t,s) g\left(s, x_n(s)\right) ds\big| \\
&\quad + \big|\int_0^1 G(t,s) g\left(s, x_n(s)\right) ds - \int_0^1 G(t,s) g\left(s, \widehat{x}(s)\right) ds\big| \\
&\leq |\widehat{x}(t) - x_n(t)| + M \int_0^1 |g\left(s, x_n(s)\right) - g\left(s, \widehat{x}(s)\right)| \, ds.
\end{aligned}$$

Let $n \to +\infty$, by the continuity of g, we deduce that $\big|\widehat{x}(t) - \int_0^1 G(t,s) g\left(s, \widehat{x}(s)\right) ds\big| = 0$. So

$$\widehat{x}(t) = \int_0^1 G(t,s) g\left(s, \widehat{x}(s)\right) ds, \quad t \in [0,1], \tag{5.6}$$

which implies that $\widehat{x} \in \Sigma$. Therefore, Σ is closed. The proof of Theorem 4.1 is complete. □

Open problem. With the assumptions of Theorem 5.1, is the set of positive solutions discrete or continuum?

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