Research Article

An Efficient Approach to Numerical Study of the MRLW Equation with B-Spline Collocation Method

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A septic B-spline collocation method is implemented to find the numerical solution of the modified regularized long wave (MRLW) equation. Three test problems including the single soliton and interaction of two and three solitons are studied to validate the proposed method by calculating the error norms L_2 and L_∞ and the invariants I_1, I_2, and I_3. Also, we have studied the Maxwellian initial condition pulse. The numerical results obtained by the method show that the present method is accurate and efficient. Results are compared with some earlier results given in the literature. A linear stability analysis of the method is also investigated.

1. Introduction

The generalized regularized long wave equation is given by

$$U_t + U_x + \delta U^p U_x - \mu U_{xxt} = 0, \tag{1}$$

where p is a positive integer and δ and μ are positive constants. This equation is one of the most important nonlinear wave equation used a model for small amplitude long waves on the surface of water in a channel [1, 2]. A few authors solved the equation numerically: among others, Zhang [3] used a finite difference method for a Cauchy problem and Kaya [4] applied the Adomian decomposition method and a quasilinearization method based on finite differences was used by Ramos [5]. Roshan [6] implemented the Petrov-Galerkin method using a linear hat function as the trial function and a quintic B-spline function as the test function. A mesh-free technique for the numerical solution of the equation has been presented by Mokhtari and Mohammadi [7]. For $p = 1$,

$$U_t + U_x + \delta U U_x - \mu U_{xxt} = 0. \tag{2}$$

Equation (2) is known as regularized long wave equation, originally introduced to describe the behavior of the undular bore by Peregrine [1] and later widely studied by Benjamin et al. [8]. The RLW equation has been solved numerically by finite element methods [9–22], finite difference methods [23–26], Fourier pseudospectral [27], and mesh-free method [28]. For $p = 2$,

$$U_t + U_x + 6U^2 U_x - \mu U_{xxt} = 0. \tag{3}$$

Another particular case of (2) is called modified regularized long wave (MRLW) equation. Like RLW equation, the MRLW equation has been solved numerically by various methods. Among many others, a collocation solution to the equation using quintic B-spline finite element method is developed by Gardner et al. [29]. Khalifa et al. [30, 31] obtained the numerical solutions of the equation using finite difference method and cubic B-spline collocation finite element method. Solutions based on collocation method with quadratic B-spline finite elements and the central finite difference method for time are investigated by Raslan [32]. The equation was solved with a collocation finite element method using quadratic, cubic, quartic, and quintic B-splines to obtain the numerical

solutions of the single solitary wave by Raslan and Hassan [33]. Haq et al. [34] have designed a numerical scheme based on quartic B-spline collocation method for the numerical solution of the equation. Ali [35] has formulated a classical radial basis functions (RBFs) collocation method for solving the equation. Karakoc et al. [36] have obtained a type of the quintic B-spline collocation procedure in which nonlinear term in the equation is linearized by using the form introduced by the Rubin and Graves [37] to solve the equation. A Petrov-Galerkin method using cubic B-spline function as trial function and a quadratic B-spline function as the test function is set up to solve the equation by Karakoc and Geyikli [38]. A homotopy analysis method has been employed to obtain approximate numerical solution of the modified regularized long wave (MRLW) equation with some specified initial conditions by Khan et al. [39].

In the present paper, a numerical scheme based on the septic B-spline collocation method has been set up for solving the MRLW equation with a variant of both initial and boundary conditions. This paper is set out as follows. In Section 2, septic B-spline collocation scheme is presented. Also stability analysis is considered. In Section 3, test problems including single, two, and three solitary waves and Maxwellian initial condition are discussed. Finally in Section 4, a summary is given at the end of the paper.

2. Septic B-Spline Finite Element Solution

Consider the MRLW Equation (3) given with the following boundary conditions,

$$
\begin{aligned}
U(a,t) = 0, \qquad U(b,t) = 0, \\
U_x(a,t) = U_x(b,t) = 0, \\
U_{xx}(a,t) = U_{xx}(b,t) = 0, \\
U_{xxx}(a,t) = U_{xxx}(b,t) = 0, \quad t > 0,
\end{aligned}
\tag{4}
$$

and the initial condition

$$
U(x,0) = f(x) \quad a \le x \le b.
\tag{5}
$$

In order to be able to apply the numerical method, the solution region of the problem is restricted over an interval $a \le x \le b$. Space interval $[a,b]$ is partitioned into uniformly sized finite elements of length h by the nodes x_m such that $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ and $h = x_{m+1} - x_m, m = -3, -2, \ldots, N+2, N+3$. The set of septic B-spline functions $\{\phi_{-3}(x), \phi_{-2}(x), \phi_{-1}(x), \ldots, \phi_{N+1}(x), \phi_{N+2}(x), \phi_{N+3}(x)\}$ forms a basis over the problem domain $[a,b]$. A global approximation $U_N(x,t)$ is expressed in terms of septic B-splines as

$$
U_N(x,t) = \sum_{i=-3}^{N+3} \phi_i(x)\, \delta_i(t),
\tag{6}
$$

where $\delta_i(t)$s are time dependent parameters to be determined from the initial, boundary, and collocation conditions.

Septic B-splines $\phi_m(x)$, $(m = -3(1)N+3)$, at the knots x_m are defined over the interval $[a,b]$ by [40]

$$
\phi_m(x) = \frac{1}{h^7}
\begin{cases}
(x - x_{m-4})^7 & [x_{m-4}, x_{m-3}] \\
(x - x_{m-4})^7 - 8(x - x_{m-3})^7 & \\
& [x_{m-3}, x_{m-2}] \\
(x - x_{m-4})^7 - 8(x - x_{m-3})^7 & \\
\quad +28(x - x_{m-2})^7 & \\
& [x_{m-2}, x_{m-1}] \\
(x - x_{m-4})^7 - 8(x - x_{m-3})^7 & \\
\quad +28(x - x_{m-2})^7 - 56(x - x_{m-1})^7 & \\
& [x_{m-1}, x_m] \\
(x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 & \\
\quad +28(x_{m+2} - x)^7 - 56(x_{m+1} - x)^7 & \\
& [x_m, x_{m+1}] \\
(x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 & \\
\quad +28(x_{m+2} - x)^7 & \\
& [x_{m+1}, x_{m+2}] \\
(x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 & \\
& [x_{m+2}, x_{m+3}] \\
(x_{m+4} - x)^7 & [x_{m+3}, x_{m+4}] \\
0 & \text{otherwise.}
\end{cases}
\tag{7}
$$

Using expansion (6) and trial function (7), the nodal values $(U_N)_m$ and their first, second, and third derivatives $(U_N)_x, (U_N)_{xx}, (U_N)_{xxx}$ can be calculated at the nodal points x_m in terms of nodal parameters δ_m by the following set of equations:

$$
\begin{aligned}
(U_N)_m &= \delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m \\
&\quad + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3}, \\
(U_{Nx})_m &= \frac{7}{h}\left(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1}\right. \\
&\quad \left. + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}\right), \\
(U_{Nxx})_m &= \frac{42}{h^2}\left(\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_m\right. \\
&\quad \left. + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3}\right), \\
(U_{Nxxx})_m &= \frac{210}{h^3}\left(-\delta_{m-3} - 8\delta_{m-2} + 19\delta_{m-1}\right. \\
&\quad \left. - 19\delta_{m+1} + 8\delta_{m+2} + \delta_{m+3}\right).
\end{aligned}
\tag{8}
$$

The splines $\phi_m(x)$ and their six principle derivatives vanish outside the interval $[x_{m-4}, x_{m+4}]$.

To apply the proposed method, Crank-Nicolson approximation for the space derivatives U_x and U_{xx} and usual first

TABLE 1: Invariants and error norms for single solitary wave with $c = 1$, $h = 0.2$, $k = 0.025$, $0 \le x \le 100$.

t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	4.4428661	3.2998227	1.4142046	0.00000000	0.00000000
1	4.4428661	3.2998085	1.4142188	0.28537793	0.16594258
2	4.4428661	3.2997808	1.4142465	0.56248008	0.31854916
3	4.4428661	3.2997573	1.4142700	0.82836630	0.45535369
4	4.4428661	3.2997415	1.4142858	1.08566992	0.58528925
5	4.44286601	3.2997313	1.4142960	1.33772774	0.71261445
6	4.4428661	3.2997248	1.4143025	1.58675627	0.83879372
7	4.4428661	3.2997207	1.4143067	1.83403948	0.96441682
8	4.4428661	3.2997180	1.4143093	2.08032250	1.08975930
9	4.4428661	3.2997162	1.4143111	2.32602024	1.21494581
10	4.4428661	3.2997151	1.4143122	2.57148152	1.34021078

TABLE 2: Errors and invariants for single solitary wave with $c = 1$, $h = 0.2$, $k = 0.025$, $0 \le x \le 100$, at $t = 10$.

Method	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Analytical	4.4428829	3.2998316	1.4142135	0	0
Present	4.4428661	3.2997151	1.4143122	2.57148	1.334021
[6]	4.44288	3.29981	1.41416	3.00533	1.68749
Cubic B-splines coll-CN [29]	4.442	3.299	1.413	16.39	9.24
Cubic B-splines coll-PA-CN+ [29]	4.440	3.296	1.411	20.3	11.2
Cubic B-splines coll [30]	4.44288	3.29983	1.41420	9.30196	5.43718
MQ [35]	4.4428829	3.29978	1.414163	3.914	2.019
IMQ [35]	4.4428611	3.29978	1.414163	3.914	2.019
IQ [35]	4.4428794	3.29978	1.414163	3.914	2.019
GA [35]	4.4428829	3.29978	1.414163	3.914	2.019
TPS [35]	4.4428821	3.29972	1.414104	4.428	2.306
Quintic B-splines coll [36]	4.4428661	3.2997108	1.4143165	2.58891	1.35164

TABLE 3: Invariants and error norms for single solitary wave with $c = 0.3$, $h = 0.1$, $k = 0.01$, $0 \le x \le 100$.

t	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$
0	3.5819531	1.3450721	0.1537217	0.0000000	0.0000000
2	3.5819531	1.3450719	0.1537219	0.0373696	0.0211791
4	3.5819531	1.3450715	0.1537223	0.0711480	0.0387624
6	3.5819531	1.3450711	0.1537227	0.1001141	0.0515117
8	3.5819531	1.3450708	0.1537231	0.1249329	0.0614203
10	3.5819531	1.3450705	0.1537234	0.1466243	0.0700260
12	3.5819531	1.3450702	0.1537236	0.1659668	0.0775889
14	3.5819531	1.3450700	0.1537238	0.1833628	0.0844911
16	3.5819531	1.3450698	0.1537240	0.2015361	0.0909663
18	3.5819531	1.3450697	0.1537241	0.2560750	0.0993420
20	3.5819531	1.3450696	0.1537243	0.3585031	0.1702101
20 [6]	3.58197	1.34508	0.153723	0.645295	0.301923
20 [30]	3.58197	1.34508	0.153723	6.06885	2.96650
20 [34]	3.581967	1.345076	0.153723	0.508927	0.222284
20 [35] MQ	3.5819665	1.3450764	0.153723	0.51498	0.22551
20 [35] IMQ	3.5819664	1.3450764	0.153723	0.51498	0.22551
20 [35] IQ	3.5819654	1.3450764	0.153723	0.51498	0.22551
20 [35] GA	3.5819665	1.3450764	0.153723	0.51498	0.22551
20 [35] TPS	3.5819663	1.3450759	0.153723	0.51498	0.26605
20 [36]	3.5820204	1.3450974	0.1537250	0.8112594	0.3569076

TABLE 4: Invariants and error norms for single solitary wave with $c = 0.6$, $h = 0.1$, $k = 0.1$, $-40 \leq x \leq 60$.

t	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$	CPU Time
4	8.070902	4.100549	14.361115	2.716636	1.323585	0.437 s
8	8.070925	4.100534	14.361111	5.240855	2.458097	0.908 s
12	8.070943	4.100519	14.361103	7.543043	3.372256	1.435 s
16	8.070949	4.100503	14.361089	9.662561	4.154427	1.774 s
20	8.070921	4.100489	14.361074	11.660256	4.862191	2.319 s

TABLE 5: Invariants and error norms for single solitary wave with $c = 0.18$, $h = 0.1$, $k = 0.1$, $-80 \leq x \leq 120$.

t	I_1	I_2	I_3	$L_2 \times 10^5$	$L_\infty \times 10^5$	CPU Time
4	7.809873	2.129887	7.031111	3.077451	1.138290	0.801 s
8	7.809875	2.129887	7.031112	6.124574	2.313797	1.571 s
12	7.809877	2.129887	7.031112	9.106266	3.455915	2.470 s
16	7.809879	2.129887	7.031113	11.967534	4.530312	3.124 s
20	7.809880	2.129887	7.031115	14.731556	5.538799	3.789 s

order forward difference formula for the time derivative of the U in (3) have been used, which lead to

$$\frac{U^{n+1} - U^n}{\Delta t} + \frac{U_x^{n+1} + U_x^n}{2} + 6\frac{\left(U^2 U_x\right)^{n+1} + \left(U^2 U_x\right)^n}{2}$$

$$-\mu\frac{U_{xx}^{n+1} - U_{xx}^n}{\Delta t} = 0. \tag{9}$$

In order to linearize the nonlinear term $(U^2 U_x)^{n+1}$, we can write the term as follows,

$$U^2 U_x = UUU_x, \tag{10}$$

and apply the linearization form introduced by Rubin and Graves [37]

$$(UU_x)^{n+1} = U^{n+1}U_x^n + U^n U_x^{n+1} - U^n U_x^n \tag{11}$$

to (9), and we get

$$U^{n+1} + \frac{\Delta t}{2}U_x^{n+1} + 3\Delta t U^{n+1}$$

$$\times \left(U^{n+1}U_x^n + U^n U_x^{n+1} - U^n U_x^n\right) - \mu U_{xx}^{n+1} \tag{12}$$

$$= U^n - \frac{\Delta t}{2}U_x^n - 3\Delta t U^n (UU_x)^n - \mu U_{xx}^n.$$

Substituting the approximate solution U_N and putting the nodal values of U and its derivatives given by (8) into (12) one obtains the following iterative system for $m = 0, 1, \ldots, N$:

$$\delta_{m-3}^{n+1}\left(1 + \alpha_1 - \alpha_2 - \alpha_3\right) + \delta_{m-2}^{n+1}\left(120 + 120\alpha_1 - 56\alpha_2 - 24\alpha_3\right)$$

$$+ \delta_{m-1}^{n+1}\left(1191 + 1191\alpha_1 - 245\alpha_2 - 15\alpha_3\right)$$

$$+ \delta_m^{n+1}\left(2416 + 2416\alpha_1 + 80\alpha_3\right)$$

$$+ \delta_{m+1}^{n+1}\left(1191 + 1191\alpha_1 + 245\alpha_2 - 15\alpha_3\right)$$

$$+ \delta_{m+2}^{n+1}\left(120 + 120\alpha_1 + 56\alpha_2 - 24\alpha_3\right)$$

$$+ \delta_{m+3}^{n+1}\left(1 + \alpha_1 + \alpha_2 - \alpha_3\right)$$

$$= \delta_{m-3}^n\left(1 - \alpha_1 + \alpha_2 - \alpha_3\right)$$

$$+ \delta_{m-2}^n\left(120 - 120\alpha_1 + 56\alpha_2 - 24\alpha_3\right)$$

$$+ \delta_{m-1}^n\left(1191 - 1191\alpha_1 + 245\alpha_2 - 15\alpha_3\right)$$

$$+ \delta_m^n\left(2416 - 2416\alpha_1 + 80\alpha_3\right)$$

$$+ \delta_{m+1}^n\left(1191 - 1191\alpha_1 - 245\alpha_2 - 15\alpha_3\right)$$

$$+ \delta_{m+2}^n\left(120 - 120\alpha_1 - 56\alpha_2 - 24\alpha_3\right)$$

$$+ \delta_{m+3}^n\left(1 - \alpha_1 - \alpha_2 - \alpha_3\right), \tag{13}$$

where

$$\alpha_1 = 3\Delta t, \qquad \alpha_2 = \frac{7\Delta t}{2h}, \qquad \alpha_3 = \frac{42\mu}{h^2}. \tag{14}$$

The newly obtained iterative system (13) consists of $N + 1$ linear equation in $N + 7$ unknowns $(\delta_{-3}, \delta_{-2}, \delta_{-1}, \delta_0, \ldots, \delta_N, \delta_{N+1}, \delta_{N+2}, \delta_{N+3})^T$. To obtain a unique solution of this system, six additional constraints are required. Applying the boundary conditions (4) and using the values of (8), these constraints are used and this enables us to eliminate the unknowns $\delta_{-3}, \delta_{-2}, \delta_{-1}, \delta_0, \ldots, \delta_N, \delta_{N+1}, \delta_{N+2}, \delta_{N+3}$ from system (13). So system (13) is reduced to a septa-diagonal system of $(N + 1)$ linear equations in $(N + 1)$ unknowns given by $A\delta^{n+1} = B\delta^n$, where $\delta^n = (\delta_0, \delta_1, \delta_2, \ldots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T$. The coefficient matrixes are given by

TABLE 6: Comparison of the invariants for the interaction of two solitary waves with results from [30] with $h = 0.2$, $k = 0.025$ in the region $0 \leq x \leq 250$.

	Present method				[30]		
t	I_1	I_2	I_3	CPU Time	I_1	I_2	I_3
0	11.4676	14.6292	22.8803	—	11.4677	14.6291	22.8806
2	11.4678	14.6282	22.8812	0.209 s	11.4677	14.6292	22.8807
4	11.4679	14.6282	22.8812	0.413 s	11.4677	14.6292	22.8807
6	11.4681	14.6181	22.8914	1.101 s	11.4677	14.6295	22.8806
8	11.4675	14.1393	23.3702	1.219 s	11.4677	14.6451	22.8454
10	11.4674	14.0502	23.4592	1.421 s	11.4677	14.5963	22.8913
12	11.4685	14.6817	22.8278	2.027 s	11.4677	14.6287	22.8814
14	11.4687	14.6649	22.8446	2.228 s	11.4677	14.6295	22.8807
16	11.4688	14.6459	22.8635	2.432 s	11.4677	14.6294	22.8808
18	11.4690	14.6370	22.8725	3.004 s	11.4677	14.6293	22.8809
20	11.4691	14.6331	22.8763	3.239 s	11.4677	14.6292	22.8809
20 [6]	11.4677	14.6299	22.8806				
20 [34]	11.4677	14.5830	22.6965				
20 [35] MQ	11.4676	14.5830	22.6965				
20 [35] IMQ	11.4676	14.5830	22.6965				
20 [35] IQ	11.4676	14.5830	22.6965				
20 [35] GA	11.4676	14.5830	22.6965				
20 [35] TPS	11.4677	14.5824	22.6942				
20 [36]	11.4691	14.6331	22.8764				

TABLE 7: Comparison of invariants for the interaction of three solitary waves with results from [30] with $h = 0.2$, $k = 0.025$ in the region $0 \leq x \leq 250$.

	Present method				[30]		
t	I_1	I_2	I_3	CPU Time	I_1	I_2	I_3
0	14.9800	15.8374	23.0081	—	13.6891	15.4549	22.8816
5	14.9804	15.7991	23.0465	0.536 s	13.6891	15.3109	22.6939
10	14.9783	14.3275	24.5180	1.442 s	13.6891	15.6514	22.8388
15	14.9809	15.5048	23.3408	2.349 s	13.6891	15.6548	22.9347
20	14.9814	15.7504	23.0952	3.251 s	13.6891	15.6557	22.9330
25	14.9819	15.8496	22.9960	4.162 s	13.6892	15.6559	22.9336
30	14.9823	15.8626	22.9829	5.063 s	13.6894	15.6559	22.9348
35	14.9828	15.8622	22.9833	5.573 s	13.6913	15.6564	22.9343
40	14.9832	15.8605	22.9851	6.481 s	13.7015	15.6566	22.9335
45	14.9461	15.8613	22.9843	7.388 s	13.7043	15.6563	22.9303
45 [34]	13.7043	15.6563	22.9303				
45 [35] MQ	14.9681	15.7343	22.5966				
45 [35] IMQ	14.9680	15.7343	22.5966				
45 [35] IQ	14.9681	15.7343	22.5966				
45 [35] GA	14.9681	15.7343	22.5966				
45 [35] TPS	14.9682	15.7337	22.5944				
45 [36]	14.7145	15.4927	23.3529				

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & \ddots \\ & & a_{m,m-3} & a_{m,m-2} & a_{m,m-1} & a_{m,m} & a_{m,m+1} & a_{m,m+2} & a_{m,m+3} \\ & & & & & \ddots \\ & & & a_{n-1,n-4} & a_{n-1,n-3} & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} \\ & & & & a_{n,n-3} & a_{n,n-2} & a_{n,n-1} & a_{n,n} & a_{n,n+1} \\ & & & & & a_{n+1,n-2} & a_{n+1,n-1} & a_{n+1,n} & a_{n+1,n+1} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ & & & & \ddots \\ & & b_{m,m-3} & b_{m,m-2} & b_{m,m-1} & b_{m,m} & b_{m,m+1} & b_{m,m+2} & b_{m,m+3} \\ & & & & & \ddots \\ & & & b_{n-1,n-4} & b_{n-1,n-3} & b_{n-1,n-2} & b_{n-1,n-1} & b_{n-1,n} & b_{n-1,n+1} \\ & & & & b_{n,n-3} & b_{n,n-2} & b_{n,n-1} & b_{n,n} & b_{n,n+1} \\ & & & & & b_{n+1,n-2} & b_{n+1,n-1} & b_{n+1,n} & b_{n+1,n+1} \end{bmatrix},$$

$$(15)$$

where

$$a_{11} = -\frac{41472}{297}\alpha_3, \qquad b_{11} = -\frac{41472}{297}\alpha_3,$$

$$a_{12} = -\frac{73224}{297}\alpha_3, \qquad b_{12} = -\frac{73224}{297}\alpha_3,$$

$$a_{13} = -\frac{20736}{297}\alpha_3, \qquad b_{13} = -\frac{20736}{297}\alpha_3,$$

$$a_{14} = -\frac{648}{297}\alpha_3, \qquad b_{14} = -\frac{648}{297}\alpha_3,$$

$$a_{21} = \frac{215411}{297} + \frac{215411}{297}\alpha_1 - \frac{11761}{297}\alpha_2 + \frac{17893}{297}\alpha_3,$$

$$b_{21} = \frac{215411}{297} - \frac{215411}{297}\alpha_1 + \frac{11761}{297}\alpha_2 + \frac{17893}{297}\alpha_3,$$

$$a_{22} = \frac{616822.5}{297} + \frac{616822.5}{297}\alpha_1 + \frac{43513.5}{297}\alpha_2 + \frac{38665.5}{297}\alpha_3,$$

$$b_{22} = \frac{616822.5}{297} - \frac{616822.5}{297}\alpha_1 - \frac{43513.5}{297}\alpha_2 + \frac{38665.5}{297}\alpha_3,$$

$$a_{23} = \frac{340284}{297} + \frac{340284}{297}\alpha_1 + \frac{78528}{297}\alpha_2 - \frac{2532}{297}\alpha_3,$$

$$b_{23} = \frac{340284}{297} - \frac{340284}{297}\alpha_1 - \frac{78528}{297}\alpha_2 - \frac{2532}{297}\alpha_3,$$

$$a_{24} = \frac{35525.5}{297} + \frac{35525.5}{297}\alpha_1 + \frac{16682.5}{297}\alpha_2 - \frac{7109.5}{297}\alpha_3,$$

$$b_{24} = \frac{35525.5}{297} + \frac{35525.5}{297}\alpha_1 + \frac{16682.5}{297}\alpha_2 - \frac{7109.5}{297}\alpha_3,$$

$$a_{25} = 1 + \alpha_1 + \alpha_2 - \alpha_3, \qquad b_{25} = 1 - \alpha_1 - \alpha_2 - \alpha_3,$$

$$a_{31} = \frac{34432}{297} + \frac{34432}{297}\alpha_1 - \frac{15424}{297}\alpha_2 - \frac{5920}{297}\alpha_3,$$

$$b_{31} = \frac{34432}{297} - \frac{34432}{297}\alpha_1 + \frac{15424}{297}\alpha_2 - \frac{5920}{297}\alpha_3,$$

$$a_{32} = \frac{352833}{297} + \frac{352833}{297}\alpha_1 - \frac{71871}{297}\alpha_2 - \frac{3561}{297}\alpha_3,$$

$$b_{32} = \frac{352833}{297} - \frac{352833}{297}\alpha_1 + \frac{71871}{297}\alpha_2 - \frac{3561}{297}\alpha_3,$$

$$a_{33} = \frac{717432}{297} + \frac{717432}{297}\alpha_1 + \frac{120}{297}\alpha_2 + \frac{23880}{297}\alpha_3,$$

$$b_{33} = \frac{717432}{297} - \frac{717432}{297}\alpha_1 - \frac{120}{297}\alpha_2 + \frac{23880}{297}\alpha_3,$$

$$a_{34} = \frac{353726}{297} + \frac{353726}{297}\alpha_1 + \frac{72766}{297}\alpha_2 - \frac{4454}{297}\alpha_3,$$

$$b_{34} = \frac{353726}{297} - \frac{353726}{297}\alpha_1 - \frac{72766}{297}\alpha_2 - \frac{4454}{297}\alpha_3,$$

$$a_{35} = 120 + 120\alpha_1 + 56\alpha_2 - 24\alpha_3,$$

$$b_{35} = 120 - 120\alpha_1 - 56\alpha_2 - 24\alpha_3,$$

$$a_{36} = 1 + \alpha_1 + \alpha_2 - \alpha_3,$$

$$b_{36} = 1 + \alpha_1 + \alpha_2 - \alpha_3,$$

$$a_{m,m-3} = 1 + \alpha_1 - \alpha_2 - \alpha_3,$$

$$b_{m,m-3} = 1 - \alpha_1 + \alpha_2 - \alpha_3,$$

$a_{m,m-2} = 120 + 120\alpha_1 - 56\alpha_2 - 24\alpha_3,$

$b_{m,m-2} = 120 - 120\alpha_1 + 56\alpha_2 - 24\alpha_3,$

$a_{m,m-1} = 1191 + 1191\alpha_1 - 245\alpha_2 - 15\alpha_3,$

$b_{m,m-1} = 1191 - 1191\alpha_1 + 245\alpha_2 - 15\alpha_3,$

$a_{m,m} = 2416 + 2416\alpha_1 + 80\alpha_3,$

$b_{m,m} = 2416 - 2416\alpha_1 + 80\alpha_3,$

$a_{m,m+1} = 1191 + 1191\alpha_1 + 245\alpha_2 - 15\alpha_3,$

$b_{m,m+1} = 1191 - 1191\alpha_1 - 245\alpha_2 - 15\alpha_3,$

$a_{m,m+2} = 120 + 120\alpha_1 + 56\alpha_2 - 24\alpha_3,$

$b_{m,m+2} = 120 - 120\alpha_1 - 56\alpha_2 - 24\alpha_3,$

$a_{m,m+3} = 1 + \alpha_1 + \alpha_2 - \alpha_3,$

$b_{m,m+3} = 1 - \alpha_1 - \alpha_2 - \alpha_3,$

$$m = 4\,(1)\,n - 2,$$

$a_{n-1,n-4} = 1 + \alpha_1 - \alpha_2 - \alpha_3,$

$b_{n-1,n-4} = 1 - \alpha_1 + \alpha_2 - \alpha_3,$

$a_{n-1,n-3} = 120 + 120\alpha_1 - 56\alpha_2 - 24\alpha_3,$

$b_{n-1,n-3} = 120 - 120\alpha_1 + 56\alpha_2 - 24\alpha_3,$

$a_{n-1,n-2} = \dfrac{353726}{297} + \dfrac{353726}{297}\alpha_1 - \dfrac{72766}{297}\alpha_2 - \dfrac{4454}{297}\alpha_3,$

$b_{n-1,n-2} = \dfrac{353726}{297} - \dfrac{353726}{297}\alpha_1 + \dfrac{72766}{297}\alpha_2 - \dfrac{4454}{297}\alpha_3,$

$a_{n-1,n-1} = \dfrac{717432}{297} + \dfrac{717432}{297}\alpha_1 - \dfrac{120}{297}\alpha_2 + \dfrac{23880}{297}\alpha_3,$

$b_{n-1,n-1} = \dfrac{717432}{297} - \dfrac{717432}{297}\alpha_1 + \dfrac{120}{297}\alpha_2 + \dfrac{23880}{297}\alpha_3,$

$a_{n-1,n} = \dfrac{352833}{297} + \dfrac{352833}{297}\alpha_1 + \dfrac{71871}{297}\alpha_2 - \dfrac{3561}{297}\alpha_3,$

$b_{n-1,n} = \dfrac{352833}{297} - \dfrac{352833}{297}\alpha_1 - \dfrac{71871}{297}\alpha_2 - \dfrac{3561}{297}\alpha_3,$

$a_{n-1,n+1} = \dfrac{34432}{297} + \dfrac{34432}{297}\alpha_1 + \dfrac{15424}{297}\alpha_2 - \dfrac{5920}{297}\alpha_3,$

$b_{n-1,n+1} = \dfrac{34432}{297} - \dfrac{34432}{297}\alpha_1 - \dfrac{15424}{297}\alpha_2 - \dfrac{5920}{297}\alpha_3,$

$a_{n,n-3} = 1 + \alpha_1 - \alpha_2 - \alpha_3, \qquad b_{n,n-3} = 1 - \alpha_1 + \alpha_2 - \alpha_3,$

$a_{n,n-2} = \dfrac{35525.5}{297} + \dfrac{35525.5}{297}\alpha_1 - \dfrac{16682.5}{297}\alpha_2 - \dfrac{7109.5}{297}\alpha_3,$

$b_{n,n-2} = \dfrac{35525.5}{297} - \dfrac{35525.5}{297}\alpha_1 + \dfrac{16682.5}{297}\alpha_2 - \dfrac{7109.5}{297}\alpha_3,$

$a_{n,n-1} = \dfrac{340284}{297} + \dfrac{340284}{297}\alpha_1 - \dfrac{78528}{297}\alpha_2 - \dfrac{2532}{297}\alpha_3,$

$b_{n,n-1} = \dfrac{340284}{297} - \dfrac{340284}{297}\alpha_1 + \dfrac{78528}{297}\alpha_2 - \dfrac{2532}{297}\alpha_3,$

$a_{n,n} = \dfrac{616822.5}{297} + \dfrac{616822.5}{297}\alpha_1 - \dfrac{43513.5}{297}\alpha_2 + \dfrac{38665.5}{297}\alpha_3,$

$b_{n,n} = \dfrac{616822.5}{297} - \dfrac{616822.5}{297}\alpha_1 + \dfrac{43513.5}{297}\alpha_2 + \dfrac{38665.5}{297}\alpha_3,$

$a_{n,n+1} = \dfrac{215411}{297} + \dfrac{215411}{297}\alpha_1 + \dfrac{11761}{297}\alpha_2 + \dfrac{17893}{297}\alpha_3,$

$b_{n,n+1} = \dfrac{215411}{297} - \dfrac{215411}{297}\alpha_1 - \dfrac{11761}{297}\alpha_2 + \dfrac{17893}{297}\alpha_3,$

$a_{n+1,n-2} = -\dfrac{648}{297}\alpha_3, \qquad b_{n+1,n-2} = -\dfrac{648}{297}\alpha_3,$

$a_{n+1,n-1} = -\dfrac{20736}{297}\alpha_3, \qquad b_{n+1,n-1} = -\dfrac{20736}{297}\alpha_3,$

$a_{n+1,n} = -\dfrac{73224}{297}\alpha_3, \qquad b_{n+1,n} = -\dfrac{73224}{297}\alpha_3,$

$a_{n+1,n+1} = -\dfrac{41472}{297}\alpha_3, \qquad b_{n+1,n+1} = -\dfrac{41472}{297}\alpha_3.$

$$(16)$$

Before starting the solution process, initial parameters δ^0 must be determined by using the initial condition and the following derivatives at the boundaries:

$$U_N(x,0) = U(x_m,0) \quad m = 0,1,2,\ldots,N,$$

$$(U_N)_x(a,0) = 0, \qquad (U_N)_x(b,0) = 0,$$

$$(U_N)_{xx}(a,0) = 0, \qquad (U_N)_{xx}(b,0) = 0,$$

$$(U_N)_{xxx}(a,0) = 0, \qquad (U_N)_{xxx}(b,0) = 0.$$

$$(17)$$

So we have the following matrix form for the initial vector δ^0:
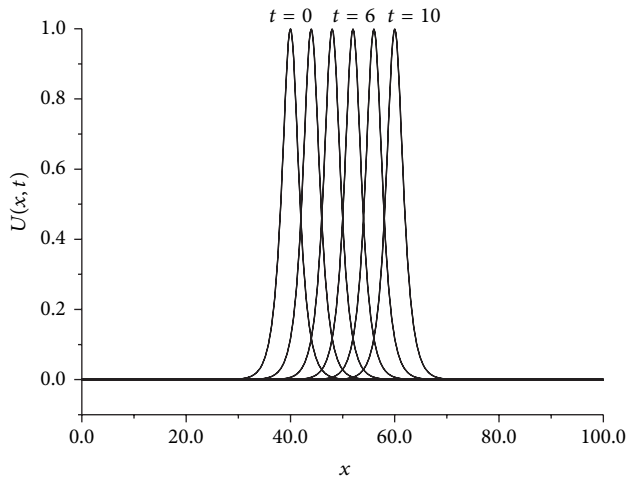
$$M\delta^0 = k, \tag{18}$$

$$M = \begin{bmatrix} 1536 & 2712 & 768 & 24 & & & & & \\ \dfrac{82731}{81} & \dfrac{210568.5}{81} & \dfrac{104796}{81} & \dfrac{10063.5}{81} & 1 & & & & \\ \dfrac{9600}{81} & \dfrac{96597}{81} & \dfrac{195768}{81} & \dfrac{96474}{81} & 120 & 1 & & & \\ & & & \ddots & & & & & \\ & & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ & & & & 1 & 120 & \dfrac{96474}{81} & \dfrac{195768}{81} & \dfrac{96597}{81} & \dfrac{9600}{81} \\ & & & & & 1 & \dfrac{10063.5}{81} & \dfrac{104796}{81} & \dfrac{210568.5}{81} & \dfrac{82731}{81} \\ & & & & & & 24 & 768 & 2712 & 1536 \end{bmatrix}, \tag{19}$$

$\delta^0 = (\delta_0, \delta_1, \delta_2, \ldots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T$, and $k = (U(x_0, 0), U(x_1, 0), \ldots, U(x_{N-1}, 0), U(x_N, 0))^T$.

2.1. A Linear Stability Analysis. We have investigated stability analysis by applying the von-Neumann approach in which the growth factor of typical Fourier mode is given by

$$\delta_j^n = \xi^n e^{ijkh}, \quad \left(i = \sqrt{-1}\right), \tag{20}$$

where k is a mode number and h is the element size. To apply this method, we have linearized the nonlinear term $U^2 U_x$ by considering U^2 as a constant such as Z_i in (9). If we substitute (20) into the iterative system (13) we obtain the following equation:

$$\xi^{n+1} = g\xi^n, \tag{21}$$

where g is the growth factor. We have identified the collocation points with the nodes and used (8) to evaluate U_m and its space derivatives in (3). This leads to a set of ordinary differential equations in the following form:

$$\begin{aligned} &\dot{\delta}_{m-3} + 120\dot{\delta}_{m-2} + 1191\dot{\delta}_{m-1} + 2416\dot{\delta}_m + 1191\dot{\delta}_{m+1} \\ &+ 120\dot{\delta}_{m+2} + \dot{\delta}_{m+3} + \frac{7}{h}(1 + 6Z_m) \\ &\times (-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} \\ &\quad + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}) \\ &- \frac{42\mu}{h^2}(\dot{\delta}_{m-3} + 24\dot{\delta}_{m-2} + 15\dot{\delta}_{m-1} - 80\dot{\delta}_m \\ &\quad + 15\dot{\delta}_{m+1} + 24\dot{\delta}_{m+2} + \dot{\delta}_{m+3}) = 0, \end{aligned} \tag{22}$$

where $Z_m = (\delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3})^2$. Here $\cdot$ denotes derivative with respect to time. If the parameters δ_i's and their time derivatives in

where

(22) are discretized by the Crank-Nicolson formula and usual forward finite difference approximation, respectively,

$$\delta_i = \frac{\delta^n + \delta^{n+1}}{2}, \qquad \dot{\delta}_i = \frac{\delta^{n+1} - \delta^n}{\Delta t}, \tag{23}$$

we obtain a recurrence relationship between two time levels n and $n+1$ relating two unknown parameters δ_i^{n+1}, δ_i^n for $i = m-3, m-2, \ldots, m+2, m+3$,

$$\begin{aligned} &\gamma_1 \delta_{m-3}^{n+1} + \gamma_2 \delta_{m-2}^{n+1} + \gamma_3 \delta_{m-1}^{n+1} + \gamma_4 \delta_m^{n+1} + \gamma_5 \delta_{m+1}^{n+1} \\ &\quad + \gamma_6 \delta_{m+2}^{n+1} + \gamma_7 \delta_{m+3}^{n+1} \\ &= \gamma_7 \delta_{m-3}^n + \gamma_6 \delta_{m-2}^n + \gamma_5 \delta_{m-1}^n \\ &\quad + \gamma_4 \delta_m^n + \gamma_3 \delta_{m+1}^n + \gamma_2 \delta_{m+2}^n + \gamma_1 \delta_{m+3}^n, \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\gamma_1 = 1 - E - M, \quad \gamma_2 = 120 - 56E - 24M, \\ &\gamma_3 = 1191 - 245E - 15M, \quad \gamma_4 = 2416 + 80M, \\ &\gamma_5 = 1191 + 245E - 15M, \quad \gamma_6 = 120 + 56E - 24M, \\ &\gamma_7 = 1 + E - M, \quad m = 0, 1, \ldots, N, \\ &E = \frac{7\Delta t}{2h}(1 + 6Z_m), \quad M = \frac{42\mu}{h^2}. \end{aligned} \tag{25}$$

Substituting the Fourier mode (20) into (24) leads to the growth factor of the form

$$\xi = \frac{a - ib}{a + ib}, \tag{26}$$

where

$$\begin{aligned} a &= 1208 + 40M + (1191 - 15M)\cos[hk] \\ &\quad + (120 - 24M)\cos[2hk] + (1 - M)\cos[3hk], \end{aligned} \tag{27}$$

$$b = 245E\sin[hk] + 56E\sin[2hk] + E\sin[3hk],$$

FIGURE 1: Single solitary wave with $c = 1, h = 0.2, \Delta t = 0.025, 0 \le x \le 100$ at times $t = 0, 2, 4, 6, 8,$ and 10.

FIGURE 2: Single solitary wave with $c = 0.3, h = 0.1, \Delta t = 0.01, 0 \le x \le 100$ at times $t = 0, 5, 10, 15,$ and 20.

$|\xi|^2 = 1$; therefore the linearized scheme is unconditionally stable.

3. Numerical Examples and Results

In this section, we have obtained numerical solution of the MRLW equation for motion of single solitary wave, interaction of two and three solitary waves, and development of the Maxwellian initial condition into solitary waves. Accuracy of the method is measured by using the following error norms:

$$L_2 = \left\| U^{exact} - U_N \right\|_2 \simeq \sqrt{h \sum_{J=1}^{N} \left| U_j^{exact} - (U_N)_j \right|^2},$$

$$L_\infty = \left\| U^{exact} - U_N \right\|_\infty \simeq \max_j \left| U_j^{exact} - (U_N)_j \right|. \quad (28)$$

The discrete conservation properties of the MRLW equation corresponding to mass, momentum, and energy are determined by finding the following three invariants [41]:

$$I_1 = \int_a^b U dx \simeq h \sum_{J=1}^{N} U_j^n,$$

$$I_2 = \int_a^b \left[U^2 + \mu(U_x)^2 \right] dx \simeq h \sum_{J=1}^{N} \left[(U_j^n)^2 + \mu(U_x)_j^n \right], \quad (29)$$

$$I_3 = \int_a^b \left(U^4 - \mu U_x^2 \right) dx \simeq h \sum_{J=1}^{N} \left[(U_j^n)^4 - \mu(U_x)_j^n \right].$$

3.1. The Motion of Single Solitary Wave. As a first problem, MRLW equation (3) is considered with the boundary conditions $U \to 0$ as $x \to \pm\infty$ and the initial condition

$$U(x, 0) = \sqrt{c} \operatorname{sech} \left[p(x - x_0) \right]. \quad (30)$$

Single solitary wave solution of the MRLW equation has an analytical solution of the form

$$U(x, t) = \sqrt{c} \operatorname{sech} \left[p(x - (c+1)t - x_0) \right], \quad (31)$$

where $p = \sqrt{c/\mu(c+1)}$ and x_0 and c are arbitrary constants. This solution corresponds to motion of single solitary wave with amplitude $\sqrt{c}$, initially centered at x_0 and with wave velocity $1 + \mu c$. For this problem the analytical values of the invariants are [29]

$$I_1 = \frac{\pi\sqrt{c}}{p}, \qquad I_2 = \frac{2c}{p} + \frac{2\mu pc}{3}, \qquad I_3 = \frac{4c^2}{3p} - \frac{2\mu pc}{3}. \quad (32)$$

For the computational work, two sets of parameters have been chosen and discussed. First of all, we have taken the parameters $c = 1, \mu = 1, h = 0.2, x_0 = 40, k = 0.025$ over the interval $[0, 100]$ to compare our results with [6, 29, 30, 35, 36]. Thus, the solitary wave has an amplitude 1.0 and the computations are done up to time $t = 10$ to obtain the invariants and error norms L_2 and L_∞. Values of the three invariants and error norms are reported in Table 1. It is clearly seen from the table that the error norms are satisfactorily small enough and the computed values of invariants are in good agreement with their analytical values $I_1 = 4.4428829, I_2 = 3.2998316,$ and $I_3 = 1.4142135$. The percentage of the relative error of the conserved quantities $I_1, I_2,$ and I_3 is calculated with respect to the conserved quantities at $t = 0$. Percentage of relative changes of $I_1, I_2,$ and I_3 is found to be $0.001 \times 10^{-3}\%, 3.261 \times 10^{-3}\%,$ and $7.609 \times 10^{-3}\%$, respectively. Thus, the quantities $I_1, I_2,$ and I_3 remain constant during the computer run. For this case in Table 2, we compare the values of the invariants and error norms obtained by using the present method and some earlier methods [6, 29, 30, 35, 36]. From the table, we observed that the error norms obtained by the present method are less than those of other methods [6, 29, 30, 35, 36]. Figure 1 shows the motion of solitary wave with $c = 1, h = 0.2, k = 0.025$

(a)

(b)
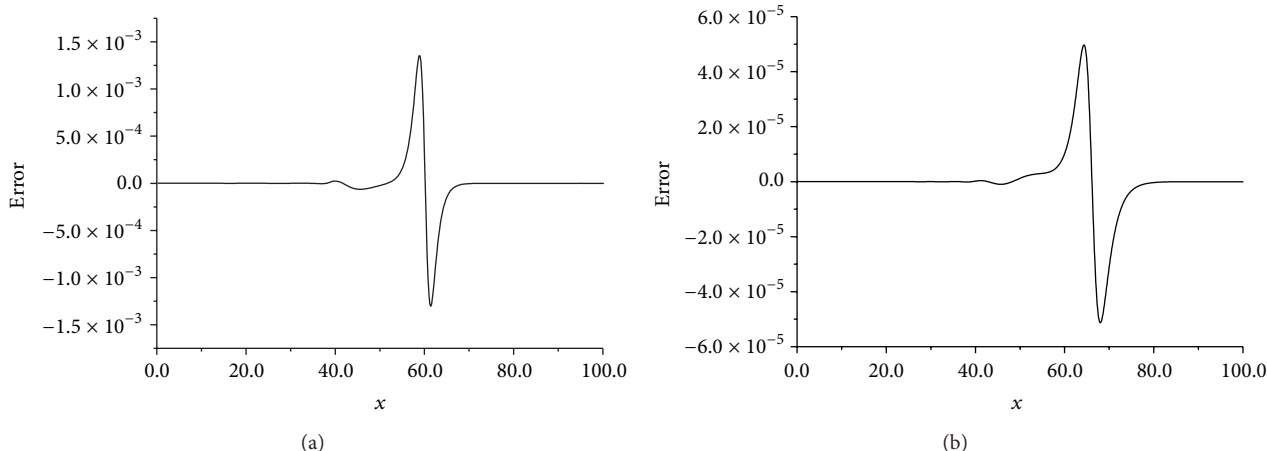
FIGURE 3: Error with (a) $c = 1, h = 0.2, \Delta t = 0.025, t = 10, 0 \leq x \leq 100$ and (b) $c = 0.3, h = 0.1, \Delta t = 0.01, t = 20, 0 \leq x \leq 100$.

TABLE 8: Invariants of MRLW equation using the Maxwellian initial condition.

t	μ	I_1	I_2	I_3	μ	I_1	I_2	I_3
0		1.7724809	1.3786628	0.7609109		1.7724809	1.2721327	0.8674410
3		1.7725930	1.4731551	0.6664186		1.7753954	1.4487372	0.6908365
6	0.1	1.7725929	1.4727329	0.6668408	0.015	1.7770146	1.4518203	0.6877534
9		1.7725935	1.4725552	0.6670185		1.7786793	1.4534252	0.6861485
12		1.7725942	1.4724619	0.6671118		1.7804784	1.4552349	0.6843388
15		1.77259499	1.4724042	0.6671695		1.7824228	1.4573443	0.6822294
0		1.7724809	1.2658662	0.8737075		1.7724809	1.3034651	0.8361087
3		1.7796208	1.4490883	0.6904854		1.7730165	1.4543782	0.6851955
6	0.01	1.7863091	1.4571676	0.6824061	0.04	1.7730562	1.4540759	0.6854978
9		1.7952456	1.4658449	0.6737288		1.7730962	1.4538204	0.6857534
12		1.8089221	1.4802019	0.6593718		1.7731365	1.4537359	0.6858378
15		1.8364987	1.5120445	0.6275292		1.7731770	1.4537234	0.6858503

at different time levels. It is observed that the solitary wave moves to the right with constant velocity and amplitude. At $t = 0$, the amplitude is 1.0 which is located at $x = 40$, while it is 0.999950 at $t = 10$ located at $x = 60$. The absolute difference in amplitudes at times $t = 0$ and $t = 10$ is found to be 5×10^{-5}, so there is a little change between the amplitudes.

Secondly, we have taken the parameters $\mu = 1, c = 0.3, h = 0.1, k = 0.01$ and $x_0 = 40$ with range $[0, 100]$ to enable comparison with [6, 30, 32, 34–36]. So the solitary wave has amplitude 0.547723. Simulations are run up to time $t = 20$. Error norms L_2 and L_∞ and conserved quantities are tabulated in Table 3, together with the results obtained in [6, 30, 32, 34–36]. These results show high degree of accuracy and efficiency of the method. The invariants I_1, I_2, and I_3 have changed by less than 0.001×10^{-3}%, 0.189×10^{-3}%, and 1.656×10^{-3}% percent, respectively. Moreover, the variation of the invariants I_2 and I_3 from the initial variants is less than 2.5×10^{-6} and 2.6×10^{-6}, respectively, whereas the changes in the invariant I_1 approach to zero throughout the interval. The perspective views of traveling solitary wave at different time levels are shown in Figure 2. The distributions of the errors at time $t = 10$ and $t = 20$ are shown graphically for

solitary waves amplitudes 1 and 0.3 in Figure 3. It is seen that the maximum errors are close to the tip of the solitary waves and between -6×10^{-3} and 6×10^{-3}, -2×10^{-4}, and 2×10^{-4}, respectively. The CPU times for an Intel(R) Core i5, 2.53 GHz, are also given in Tables 4 and 5.

3.2. Interaction of Two Solitary Waves. As a second problem, interaction of two well separated solitary waves having different amplitudes and traveling in the same direction is considered by using the initial condition

$$ U(x, 0) = \sum_{j=1}^{2} A_j \text{sech} \left(p_j \left[x - x_j \right] \right), \qquad (33) $$

where $A_j = \sqrt{c_j}$, $p_j = \sqrt{c_j / \mu(c_j + 1)}$, $j = 1, 2$, and c_j and x_j are arbitrary constants. The analytical values of the invariants can be found as [29]

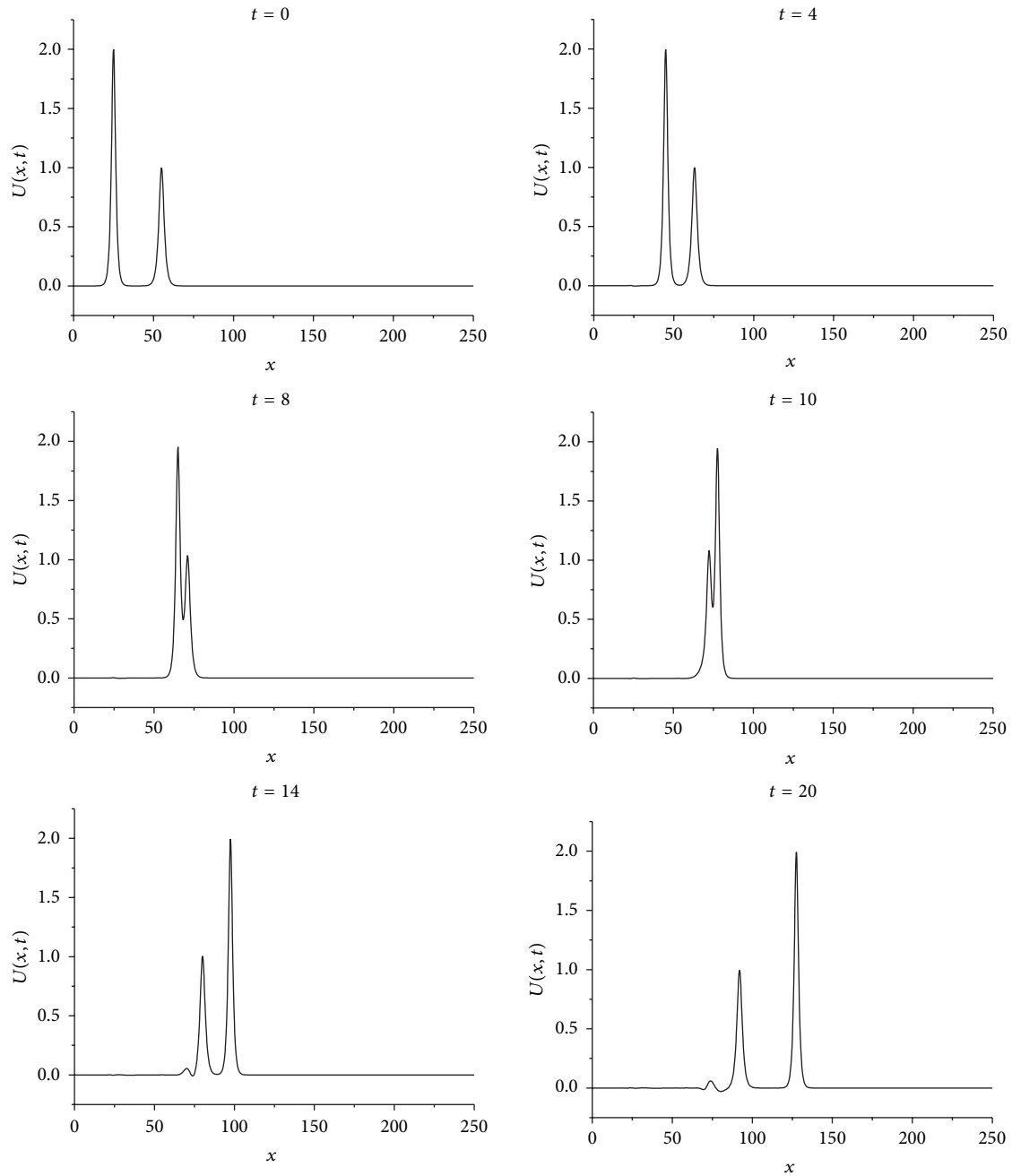$$ I_1 = \sum_{j=1}^{2} \frac{\pi \sqrt{c_j}}{p_j} = 11.467698, $$

FIGURE 4: Interaction of two solitary waves with $t = 0, 4, 8, 10, 14,$ and 20.

$$I_2 = \sum_{j=1}^{2} \left(\frac{2c_j}{p_j} + \frac{2\mu p_j c_j}{3} \right) = 14.629243,$$

$$I_3 = \sum_{j=1}^{2} \left(\frac{4c_j^2}{3p_j} - \frac{2\mu p_j c_j}{3} \right) = 22.880466.$$

$$(34)$$

We have studied the interaction of two positive solitary waves having the parameters $\mu = 1, h = 0.2, k = 0.025, c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55$ through the interval $0 \leq x \leq 250$

to coincide with those used by [6, 30, 34–36]. The simulations are maintained up to $t = 20$. Constant values $I_1, I_2,$ and I_3 at various time steps together with equivalent results for the previous methods are shown in Table 6. It is seen that the numerical values of the invariants remain almost constant during the computer run. The interaction of two solitary waves is shown in Figure 4. It can be seen from the figure that, at $t = 0$, the wave with larger amplitude is to the left of the second wave with smaller amplitude. Since the taller wave moves faster than the shorter one, it catches up and collides with the shorter one at $t = 8$ and then moves away from the
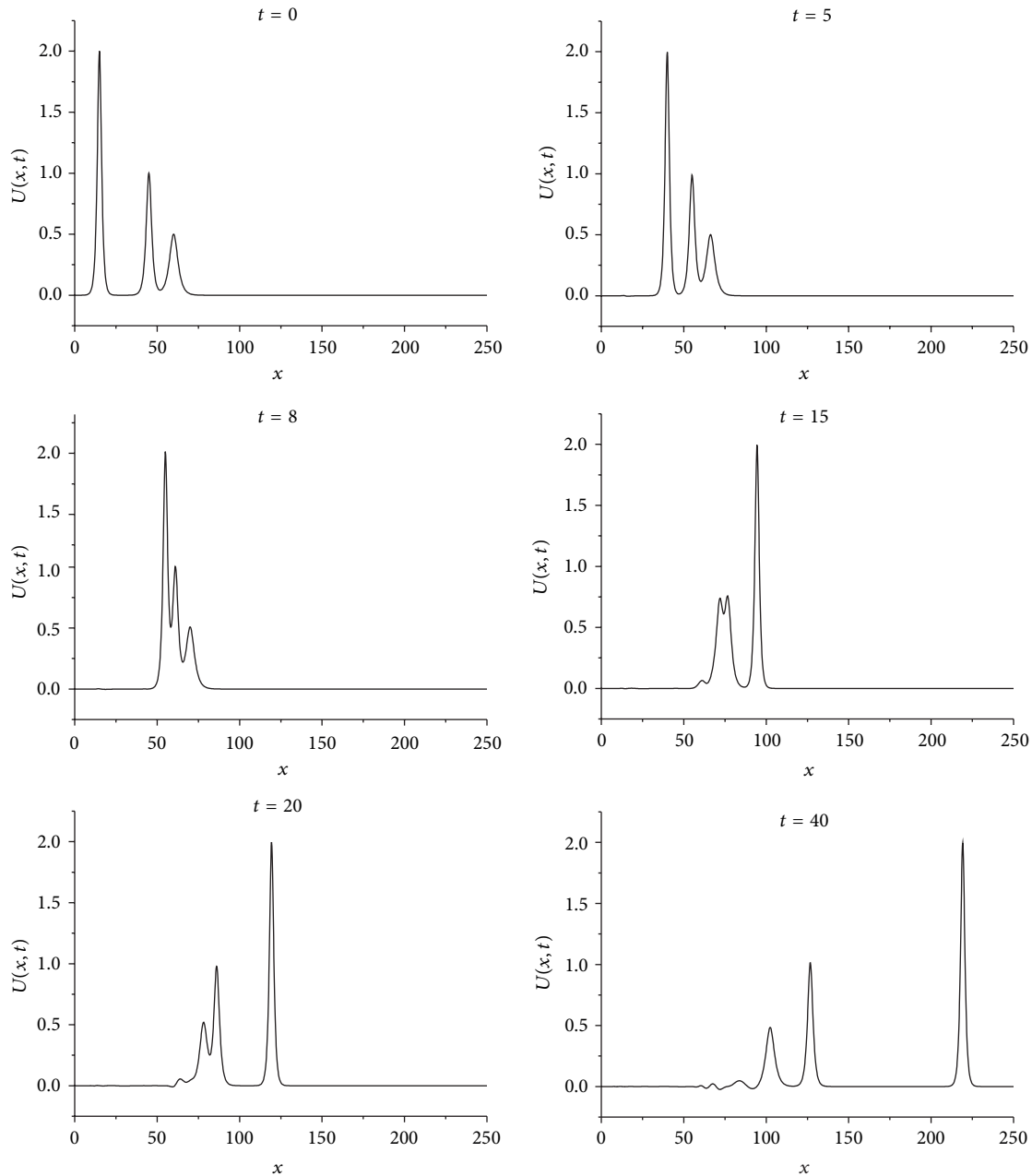
FIGURE 5: Interaction of three solitary waves with $t = 0, 5, 8, 15, 20,$ and 40.

shorter one as time increases. At $t = 20$, the amplitude of larger waves is 2.001102 at the point $x = 127.4$, whereas the amplitude of the smaller one is 0.996403 at the point $x = 92$. It is found that the absolute difference in amplitude is 3.59×10^{-3} for the smaller wave and 1.10×10^{-3} for the larger wave for this case.

3.3. Interaction of Three Solitary Waves. As a third problem, interaction of three solitary waves having different amplitudes and travelling in the same direction is studied. We consider (3) with initial conditions given by

$$U(x, 0) = \sum_{j=1}^{3} A_j \mathrm{sech}\left(p_j \left[x - x_j\right]\right), \tag{35}$$

where $A_j = \sqrt{c_j}$, $p_j = \sqrt{c_j/\mu(c_j + 1)}$, $j = 1, 2, 3$, c_j and x_j are arbitrary constants. For this problem the analytical values of the invariants are found from (32) as

$$I_1 = \sum_{j=1}^{3} \frac{\pi \sqrt{c_j}}{p_j} = 14.9801,$$

$$I_2 = \sum_{j=1}^{3} \left(\frac{2c_j}{p_j} + \frac{2\mu p_j c_j}{3}\right) = 15.8218, \tag{36}$$

$$I_3 = \sum_{j=1}^{3} \left(\frac{4c_j^2}{3p_j} - \frac{2\mu p_j c_j}{3}\right) = 22.9923.$$

(a)

(b)

(c)

(d)
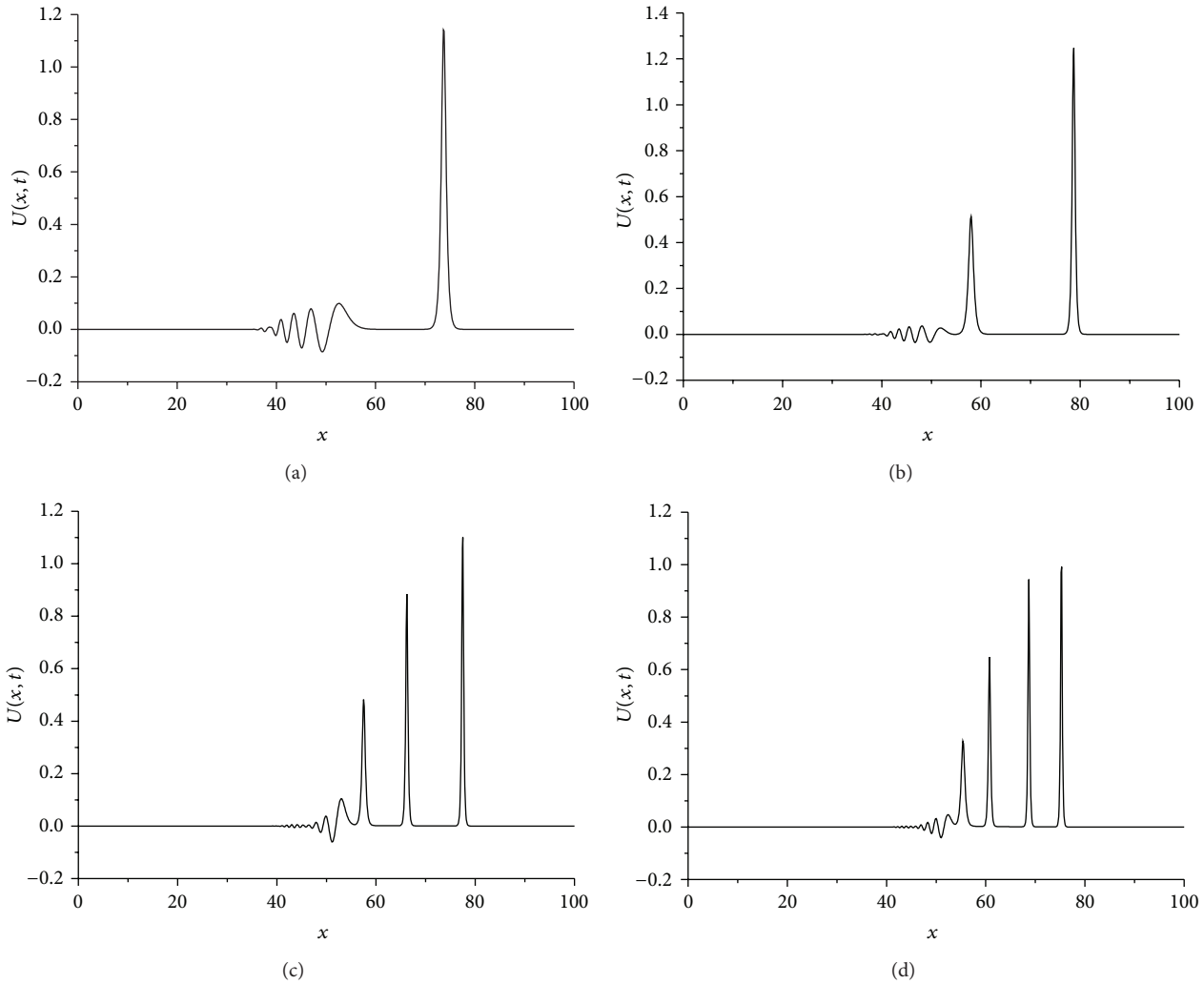
FIGURE 6: Maxwellian initial condition at $t = 14.5$ with (a) $\mu = 0.1$, (b) $\mu = 0.015$, (c) $\mu = 0.01$, and (d) $\mu = 0.04$.

In order to be able to compare with the previous works, computations are performed for the parameters $\mu = 1, h = 0.2, k = 0.025, c_1 = 4, c_2 = 1, c_3 = 0.25, x_1 = 15, x_2 = 45, x_3 = 60$ over the interval $0 \leq x \leq 250$. The simulation is run from $t = 0$ to $t = 45$. Table 7 shows a comparison of the values of the invariants obtained by the present method with those obtained in [30, 34–36]. It is evident from the table that the obtained values of the invariants remain almost constant during the computer run which are all in good agreement with their analytical values given by (36). Figure 5 depicts the interaction of three solitary waves at different times. It is observed from Figure 5 that interaction started about time $t = 10$, overlapping processes occurred between time $t = 15$ and $t = 40$, and waves started to resume their original shapes after the time $t = 40$.

3.4. The Maxwellian Initial Condition. As our last problem, the development of the Maxwellian initial condition,

$$U(x, 0) = \exp\left(-(x - 40)^2\right), \tag{37}$$

into a train of solitary waves is considered. For the Maxwellian initial condition, behavior of the solution depends on the values of μ. So we take $\mu = 0.1, \mu = 0.04, \mu = 0.015,$ and $\mu = 0.01$. The numerical values of the invariants quantities during the simulations are given in Table 8. For $\mu = 0.1$, only a single soliton occurred as depicted in Figure 6(a). When $\mu = 0.015$ and $\mu = 0.01$, two and three stable solitons occurred, respectively, as depicted in Figures 6(b) and 6(c). For $\mu = 0.04$, Maxwellian initial condition has decayed into four solitary waves as depicted in Figure 6(d). As is seen, when μ is reduced, more solitary waves occurred. All figures are drawn up at time $t = 14.5$. The peaks of the well developed wave lie on a straight line so that their velocities are linearly dependent on their amplitudes and also we observe a small oscillating tail appearing behind the last wave in all Maxwellian figures.

4. Conclusion

In this paper, a numerical treatment of the MRLW equation has been introduced using septic B-spline collocation finite

element method. The nonlinear term in the equation has been linearized by using a form given in the paper [37]. To examine the performance of the scheme, four test problems have been studied. The performance and accuracy of the method have been tested by calculating the error norms L_2 and L_∞ and the invariant quantities I_1, I_2, and I_3. Linear stability analysis proved that the present scheme is unconditionally stable. The experimental results of the algorithm are much satisfactory in comparison with the previous results. Our method can successfully be used to model the motion and interaction of the solitary waves. Thus, we can assert that our scheme is efficient and reliable for obtaining the numerical solutions of the other physically important nonlinear partial differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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