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TRIPLE POSITIVE SOLUTIONS FOR m-POINT BOUNDARY-VALUE PROBLEMS OF DYNAMIC EQUATIONS ON TIME SCALES WITH p-LAPLACIAN

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ABSTRACT. In this article we study the existence of positive solutions for m-point dynamic equation on time scales with p-Laplacian. We prove that the boundary-value problem has at least three positive solutions by applying the five functionals fixed-point theorem. An example demonstrates the main results.

1. INTRODUCTION

In recent years, dynamic equations on time scales have found a considerable interest and attracted many researchers; see for example [1, 2, 3, 4, 9, 10, 11, 12, 17, 20, 27]. The reasons seem to be two-fold. Theoretically, dynamic equations on time scales can not only unify differential and difference equations [14], but also have displayed much more complicated dynamics [7, 8, 16]. Moreover, the study of time scales has led to several important applications in the study of insect population models, neural networks, stock market, heat transfer, wound healing and epidemic models; see for example [15, 21, 23].

In this paper, we study the existence of positive solutions of m-point p-Laplacian equation on time scales

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + g(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}}, \tag{1.1}$$

with the boundary conditions

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u^{\Delta}(T) = 0, \tag{1.2}$$

or

$$u^{\Delta}(0) = 0, \quad u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.3}$$

where $\phi_p(s)$ is p-Laplacian operator; i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, with $(\phi_p)^{-1} = \phi_q$and$\frac{1}{p} + \frac{1}{q} = 1$ and $\xi_i \in (0,T)_{\mathbb{T}}$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < T$ and $a_i, b_i \in [0,\infty)$ satisfy $1 - \sum_{i=1}^{m-2} a_i \neq 0$ and $1 - \sum_{i=1}^{m-2} b_i \neq 0$ $(i = 1, 2, \dots, m-2)$.

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Some basic knowledge and definitions about time scales, which can be found in [7,8], will be used here. By using the five functionals fixed-point theorem, we prove that the boundary-value problems (1.1) (1.2) and (1.1) (1.3) has at least three positive solutions.

Throughout this paper, we assume that the following conditions are satisfied:

- (H1) $f : \mathbb{R} \to \mathbb{R}^+$ is continuous, and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$;
- (H2) $g : \mathbb{T} \to \mathbb{R}^+$ is left dense continuous ($g \in C_{ld}(\mathbb{T}, \mathbb{R}^+)$), and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$.

Recently, the boundary-value problems with p-Laplacian in the continuous case have been studied extensively in the literature; see for example [6, 13, 18, 19, 24, 25, 28]. However, to the best of our knowledge, there are not many results concerning p-Laplacian dynamic equations on time scales, see [4, 22, 26].

Zhao et al [28] studied the existence of at least three positive solutions to the following p-Laplacian problem,

$$\begin{gathered}(\phi_p(u'(t)))' + a(t)f(u,u') = 0, \quad t \in [0,1],\\ u'(0) = u(1) = 0.\end{gathered}$$

To show their main results, they applied Leggett-Williams fixed-point theorem.

Anderson et al [4] considered the following BVP on time scales:

$$\begin{gathered}[\phi_p(u^{\Delta}(t))]^{\nabla} + c(t)f(u(t)) = 0, \quad t \in (a,b)_{\mathbb{T}},\\ u(a) - B_0(u^{\Delta}(\upsilon)) = 0, \quad u^{\Delta}(b) = 0,\end{gathered}$$

where $\upsilon \in (a,b)_{\mathbb{T}}$, $f \in C_{ld}([0,+\infty),[0,+\infty))$, $c \in C_{ld}([a,b],[0,+\infty))$, and $K_m x \leq B_0(x) \leq K_M x$ for some positive constants K_m, K_M. By using a fixed-point theorem, they established the existence result for at least one positive solution.

Wang [26] studied existence criteria of three positive solutions to the following boundary-value problems for p-Laplacian dynamic equations on time scales

$$\begin{gathered}[\phi_p(u^{\Delta}(t))]^{\nabla} + a(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}},\\ u^{\Delta}(0) = 0, \quad u(T) + B_1(u^{\Delta}(\eta)) = 0, \quad \text{or}\\ u(0) - B_0(u^{\Delta}(\eta)) = 0, \quad u^{\Delta}(T) = 0.\end{gathered}$$

The main tool used in [26] is the Leggett-Williams fixed-point theorem.

Motivated by the results mentioned above, we consider the existence of solutions to (1.1) (1.2) and (1.1) (1.3). Our main results will depend on an application of the five functionals fixed-point theorem.

2. Preliminaries

In this section, we provide some background materials from theory of cones in Banach spaces, and we then state the five functionals fixed-point theorem for a cone preserving operator.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

- (i) If $u \in P$ and $\lambda \geq 0$, then $\lambda u \in P$;
- (ii) If $u \in P$ and $-u \in P$, then $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. Given a cone P in a real Banach space E, a functional $\psi : P \to \mathbb{R}$ is said to be increasing on P, provided $\psi(x) \leq \psi(y)$ for all $x, y \in P$ with $x \leq y$.

Definition 2.3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha :\to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ, β, θ be nonnegative continuous convex functionals on P, and α, ψ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers h, a, b, d and c, we define the following convex sets,

$$\begin{gathered} P(\gamma, c) = \{u \in P : \gamma(u) < c\}, \\ P(\gamma, \alpha, a, c) = \{u \in P : a \leq \alpha(u), \gamma(u) \leq c\}, \\ Q(\gamma, \beta, d, c) = \{u \in P : \beta(u) \leq d, \gamma(u) \leq c\}, \\ P(\gamma, \theta, \alpha, a, b, c) = \{u \in P : a \leq \alpha(u), \theta(u) \leq b, \gamma(u) \leq c\}, \\ Q(\gamma, \beta, \psi, h, d, c) = \{u \in P : h \leq \psi(u), \beta(u) \leq d, \gamma(u) \leq c\}. \end{gathered}$$

The following five functionals fixed-point theorem will play an important role in the proof of our main results.

Theorem 2.4 ([5]). *Let P be a cone in a real Banach space E. Suppose there exist positive numbers c and M, nonnegative continuous concave functionals α and ψ on P, and nonnegative continuous convex functionals γ, β and θ on P with*

$$\alpha(u) \leq \beta(u), \quad \|u\| \leq M\gamma(u)$$

for all $u \in \overline{P(\gamma, c)}$. Suppose that $F : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is a completely continuous operator and that there exist nonnegative numbers h, a, k, b, with $0 < a < b$ such that:

- (i) *$\{u \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(u) > b\} \neq \emptyset$ and $\alpha(Fu) > b$ for $u \in P(\gamma, \theta, \alpha, b, k, c)$;*
- (ii) *$\{u \in Q(\gamma, \beta, \psi, h, a, c) : \beta(u) < a\} \neq \emptyset$ and $\beta(Fu) < a$ for $u \in Q(\gamma, \beta, \psi, h, a, c)$;*
- (iii) *$\alpha(Fu) > b$ for $u \in P(\gamma, \alpha, b, c)$ with $\theta(Fu) > k$;*
- (iv) *$\beta(Fu) < a$ for $u \in Q(\gamma, \beta, a, c)$ with $\psi(Fu) < h$.*

Then F has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that $\beta(u_1) < a$, $b < \alpha(u_2)$ and $a < \beta(u_3)$, with $\alpha(u_3) < b$.

3. Existence of three positive solutions

In this section, by using the five functionals fixed-point theorem, we will find the existence of at least three positive solutions (1.1) (1.2) and (1.1) (1.3).

Let the Banach space $E = C_{ld}([0, T]_{\mathbb{T}}, \mathbb{R})$ with norm $\|u\| = \sup_{t \in [0,T]_{\mathbb{T}}} |u(t)|$, and define the cone, $P \subset E$, by

$$P = \{u \in E : u^{\Delta}(T) = 0, u \text{ is concave and nonnegative on } [0, T]_{\mathbb{T}}\}.$$

Suppose that there exists $l \in \mathbb{T}$ such that $\xi_{m-2} < l < T$ and $\int_l^T g(r)\nabla r > 0$ hold, then we will use the following lemma.

Lemma 3.1. *If $u \in P$, then*
 (i) *$u(t) \geq \frac{t}{T}\|u\|$ for $t \in [0,T]_{\mathbb{T}}$;*
 (ii) *$su(t) \geq tu(s)$ for $t, s \in [0,T]_{\mathbb{T}}$, with $t \leq s$.*

Proof. (i) Since $u^{\Delta\nabla}(t) \leq 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Thus, for $0 < t < T$,

$$u(t) - u(0) = \int_0^t u^{\Delta}(s)\Delta s \geq tu^{\Delta}(t),$$
$$u(T) - u(t) = \int_t^T u^{\Delta}(s)\Delta s \leq (T-t)u^{\Delta}(t)$$

from which we have

$$u(t) \geq \frac{tu(T) + (T-t)u(0)}{T} \geq \frac{t}{T}u(T) = \frac{t}{T}\|u\|.$$

(ii) If $t = s$, then the conclusion of (ii) holds. If $t < s$ with $t, s \in [0,T]_{\mathbb{T}}$, setting $x(t) = u(t) - \frac{t}{s}u(s)$, for $u \in P$, we have

$$x^{\Delta\nabla}(t) = u^{\Delta\nabla}(t) \leq 0, x(0) = u(0) \geq 0, x(s) = 0.$$

Therefore, the concavity of x implies that $x(t) \geq 0, t \in [0,s)_{\mathbb{T}}$; i.e., $su(t) > tu(s)$, for $t < s$ with $t, s \in [0,T]_{\mathbb{T}}$. This completes the proof. $\square$

We define the nonnegative, continuous concave functionals α, ψ and nonnegative continuous convex functionals β, θ, γ on the cone P by

$$\gamma(u) = \theta(u) := \max_{t \in [0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}),$$
$$\alpha(u) := \min_{t \in [l,T]_{\mathbb{T}}} u(t) = u(l),$$
$$\beta(u) := \max_{t \in [0,l]_{\mathbb{T}}} u(t) = u(l),$$
$$\psi(u) := \min_{t \in [\xi_{m-2},T]_{\mathbb{T}}} u(t) = u(\xi_{m-2}).$$

We see that, for all $u \in P$,

$$\alpha(u) = u(l) = \beta(u).$$

For notational convenience, we define

$$M = \Big(\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1 - \sum_{i=1}^{m-2} a_i} + \xi_{m-2}\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big),$$
$$m = \Big(\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1 - \sum_{i=1}^{m-2} a_i} + l\Big)\phi_q\Big(\int_l^T g(r)\nabla r\Big),$$
$$\lambda_l = \Big(\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1 - \sum_{i=1}^{m-2} a_i} + l\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big).$$

We note that $u(t)$ is a solution of (1.1) and (1.2), if and only if

$$u(t) = \frac{\sum_{i=1}^{m-2} a_i\Big(\int_0^{\xi_i} \phi_q\Big(\int_s^T g(r)f(u(r))\nabla r\Big)\Delta s\Big)}{1 - \sum_{i=1}^{m-2} a_i}$$

$$+ \int_0^t \phi_q\Big(\int_s^T g(r)f(u(r))\nabla r\Big)\Delta s, \quad t \in [0,T]_{\mathbb{T}}.$$

Theorem 3.2. *Let $0 < a < lb/T < (l\xi_{m-2}c)/T^2$, $Mb < mc$, and suppose that f satisfies the following conditions:*

- (C1) *$f(w) < \phi_p(\frac{c}{M})$, for all $0 \le w \le Tc/\xi_{m-2}$;*
- (C2) *$f(w) > \phi_p(\frac{b}{m})$, for all $b \le w \le T^2b/\xi_{m-2}^2$;*
- (C3) *$f(w) < \phi_p(\frac{a}{\lambda_l})$, for all $0 \le w \le Ta/l$.*

Then, there exist at least three positive solutions u_1, u_2, u_3 of (1.1) *and* (1.2) *such that*

$$\max_{t\in[0,l]_{\mathbb{T}}} u_1(t) < a, \quad b < \min_{t\in[l,T]_{\mathbb{T}}} u_2(t) \quad \text{and} \quad a < \max_{t\in[0,l]_{\mathbb{T}}} u_3(t) \quad \text{with} \quad \min_{t\in[l,T]_{\mathbb{T}}} u_3(t) < b.$$

Proof. Defining a completely continuous integral operator $F : P \to E$ by

$$\begin{aligned}(Fu)(t) = &\frac{\sum_{i=1}^{m-2} a_i\big(\int_0^{\xi_i} \phi_q\big(\int_s^T g(r)f(u(r))\nabla r\big)\Delta s\big)}{1-\sum_{i=1}^{m-2} a_i} \\ &+ \int_0^t \phi_q\Big(\int_s^T g(r)f(u(r))\nabla r\Big)\Delta s, \quad u \in P,\end{aligned} \tag{3.1}$$

for $t \in [0,T]_{\mathbb{T}}$, we will search for fixed points of F in the cone P. We note that, if $u \in P$, then $(Fu)(t) \ge 0$ for $t \in [0,T]_{\mathbb{T}}$, and

$$(Fu)^{\Delta}(t) = \phi_q\Big(\int_t^T g(r)f(u(r))\nabla r\Big), \quad u \in P,\ t \in [0,T]_{\mathbb{T}^\kappa}.$$

We see that $(Fu)^{\Delta}(t)$ is continuous and nonincreasing on $[0,T]_{\mathbb{T}^\kappa}$ and, $(Fu)^{\Delta\nabla}(t) \le 0$ for $[0,T]_{\mathbb{T}^\kappa\cap\mathbb{T}_\kappa}$. In addition, $(Fu)^{\Delta}(T) = 0$. This implies that $Fu \in P$, and therefore $F : P \to P$.

If $u \in \overline{P(\gamma,c)}$, then

$$\gamma(u) = \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}) = c.$$

Consequently, $0 \le u(t) \le c$ for $t \in [0,\xi_{m-2}]_{\mathbb{T}}$. By Lemma 3.1, we have

$$\|u\| \le \frac{Tu(\xi_{m-2})}{\xi_{m-2}} \le \frac{Tc}{\xi_{m-2}}.$$

This implies $0 \le u(t) \le \frac{Tc}{\xi_{m-2}}$ for $t \in [0,T]_{\mathbb{T}}$.

It follows from (C1) of Theorem 3.2 that

$$\begin{aligned}\gamma(F(u)) &= (Fu)(\xi_{m-2}) \\ &= \frac{\sum_{i=1}^{m-2} a_i\left(\int_0^{\xi_i} \phi_q\left(\int_s^T g(r)f(u(r))\nabla r\right)\Delta s\right)}{1-\sum_{i=1}^{m-2} a_i} \\ &\quad + \int_0^{\xi_{m-2}} \phi_q\Big(\int_s^T g(r)f(u(r))\nabla r\Big)\Delta s \\ &< \frac{\sum_{i=1}^{m-2} a_i\left(\int_0^{\xi_i} \phi_q\left(\int_s^T g(r)f(u(r))\nabla r\right)\Delta s\right)}{1-\sum_{i=1}^{m-2} a_i} \\ &\quad + \xi_{m-2}\phi_q\Big(\int_0^T g(r)f(u(r))\nabla r\Big)\end{aligned}$$

$$
\begin{aligned}
&< \frac{\sum_{i=1}^{m-2} a_i \left(\int_0^{\xi_i} \phi_q \left(\int_0^T g(r)f(u(r))\nabla r\right)\Delta s\right)}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad + \xi_{m-2}\phi_q\Big(\int_0^T g(r)f(u(r))\nabla r\Big) \\
&< \frac{c}{M}\Big(\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i} + \xi_{m-2}\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big) = c.
\end{aligned}
$$

So $F(u) \in \overline{P(\gamma, c)}$.

By Lemma 3.1, we obtain $\gamma(u) = u(\xi_{m-2}) \geq \frac{\xi_{m-2}}{T}\|u\|$, hence

$$
\|u\| \leq \frac{Tu(\xi_{m-2})}{\xi_{m-2}} = \frac{T\gamma(u)}{\xi_{m-2}} \text{for all } u \in P.
$$

Now we prove that (i)-(iv) of Theorem 2.4 are satisfied. Firstly, if $u \equiv \frac{Tb}{\xi_{m-2}}$, $k = \frac{Tb}{\xi_{m-2}}$, then

$$
\alpha(u) = u(l) = \frac{Tb}{\xi_{m-2}} > b, \quad \theta(u) = u(\xi_{m-2}) = \frac{Tb}{\xi_{m-2}} = k, \quad \gamma(u) = \frac{Tb}{\xi_{m-2}} < c,
$$

which show that

$$
\{u \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(u) > b\} \neq \emptyset.
$$

For $u \in P(\gamma, \theta, \alpha, b, \frac{Tb}{\xi_{m-2}}, c)$, we obtain

$$
\theta(u) = \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}) \leq \frac{Tb}{\xi_{m-2}}, \quad \alpha(u) = \min_{t\in[l,T]_{\mathbb{T}}} u(t) = u(l) \geq b,
$$

which imply

$$
0 \leq u(t) \leq \frac{Tb}{\xi_{m-2}} \quad \text{for all } t \in [0, \xi_{m-2}]_{\mathbb{T}},
$$

and $b \leq u(t)$ for all $t \in [l, T]_{\mathbb{T}}$. By Lemma 3.1, we obtain

$$
\|u\| \leq \frac{Tu(\xi_{m-2})}{\xi_{m-2}} \leq \frac{T^2 b}{\xi_{m-2}^2},
$$

as a result,

$$
b \leq u(t) \leq \frac{T^2 b}{\xi_{m-2}^2} \quad \text{for all } t \in [l, T]_{\mathbb{T}}.
$$

By (C2) of Theorem 3.2, we find

$$
\begin{aligned}
\alpha(F(u)) &= (Fu)(l) \\
&= \frac{\sum_{i=1}^{m-2} a_i \left(\int_0^{\xi_i} \phi_q \left(\int_s^T g(r)f(u(r))\nabla r\right)\Delta s\right)}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad + \int_0^l \phi_q\Big(\int_s^T g(r)f(u(r))\nabla r\Big)\Delta s \\
&> \frac{\sum_{i=1}^{m-2} a_i \left(\int_0^{\xi_i} \phi_q \left(\int_l^T g(r)f(u(r))\nabla r\right)\Delta s\right)}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad + l\phi_q\Big(\int_l^T g(r)f(u(r))\nabla r\Big)
\end{aligned}
$$

$$> \frac{b}{m}\Big(\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i} + l\Big)\phi_q\Big(\int_l^T g(r)\nabla r\Big) = b.$$

Therefore, (i) of Theorem 2.4 is satisfied.

Secondly, we show that (ii) of Theorem 2.4 is satisfied. Let $u = \frac{a\xi_{m-2}}{T}$ and $h = \frac{a\xi_{m-2}}{T}$, then

$$\gamma(u) = u(\xi_{m-2}) = \frac{a\xi_{m-2}}{T} < c, \quad \beta(u) = u(l) = \frac{a\xi_{m-2}}{T} < a,$$
$$\psi(u) = u(\xi_{m-2}) = \frac{a\xi_{m-2}}{T} = h.$$

Thus

$$\{u \in Q(\gamma, \beta, \psi, h, a, c) : \beta(u) < a\} \neq \emptyset.$$

If $u \in Q(\gamma, \beta, \psi, \frac{a\xi_{m-2}}{T}, a, c)$, then

$$\beta(u) := \max_{t\in[0,l]_{\mathbb{T}}} u(t) = u(l) \le a,$$

as a result $0 \le u(t) \le a$ for $t \in [0, l]_{\mathbb{T}}$. By Lemma 3.1,

$$\|u\| \le \frac{Tu(l)}{l} \le \frac{Ta}{l} \quad \text{for } t \in [0, T]_{\mathbb{T}},$$

hence $0 \le u(t) \le Ta/l$ for $t \in [0, T]_{\mathbb{T}}$. By (C3) of Theorem 3.2, we obtain

$$\begin{aligned}
\beta(F(u)) &= (Fu)(l)\\
&= \frac{\sum_{i=1}^{m-2} a_i \left(\int_0^{\xi_i} \phi_q\Big(\int_s^T g(r)f(u(r))\nabla r\Big)\Delta s\right)}{1-\sum_{i=1}^{m-2} a_i}\\
&\quad + \int_0^l \phi_q\left(\int_s^T g(r)f(u(r))\nabla r\right)\Delta s\\
&< \frac{\sum_{i=1}^{m-2} a_i\xi_i\phi_q\left(\int_0^T g(r)f(u(r))\nabla r\right)}{1-\sum_{i=1}^{m-2} a_i}\\
&\quad + l\phi_q\Big(\int_0^T g(r)f(u(r))\nabla r\Big)\\
&< \frac{a}{\lambda_l}\Big(\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i} + l\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big) = a.
\end{aligned}$$

Thirdly, we verify that (ii) of Theorem 2.4 is satisfied. If

$$u \in P(\gamma, \alpha, b, c) \quad \text{and} \quad \theta(F(u)) = F(u(\xi_{m-2})) > k = \frac{Tb}{\xi_{m-2}},$$

then

$$\alpha(F(u)) = (Fu)(l) \ge \frac{l}{T}F(u(l)) \ge \frac{l}{T}F(u(\xi_{m-2})) > \frac{lb}{\xi_{m-2}} > b.$$

Lastly, if

$$u \in Q(\gamma, \beta, a, c) \quad \text{and} \quad \psi(F(u)) = F(u(\xi_{m-2})) < h = \frac{a\xi_{m-2}}{T},$$

then by Lemma 3.1 we find

$$\beta(F(u)) = (Fu)(l) \le \frac{T}{l} F(u(l)) \le \frac{T}{\xi_{m-2}} F(u(\xi_{m-2})) < a$$

which shows that condition (iv) of Theorem 2.4 is satisfied.

Hence, all the conditions in Theorem 2.4 are fulfilled, therefore the boundary-value problems (1.1) and (1.2) has at least three positive solutions u_1, u_2, u_3 such that

$$\max_{t\in[0,l]_{\mathbb{T}}} u_1(t) < a, \quad b < \min_{t\in[l,T]_{\mathbb{T}}} u_2(t), \quad a < \max_{t\in[0,l]_{\mathbb{T}}} u_3(t) \quad \text{with} \quad \min_{t\in[l,T]_{\mathbb{T}}} u_3(t) < b.$$

The proof of Theorem 3.2 is complete. □

Now, we apply the five functionals fixed-point theorem to establish the existence of at least three positive solutions of (1.1) and (1.3).

We define the cone, $P_1 \subset E$, by

$$P_1 = \{u \in E : u^{\Delta}(0) = 0, u \text{ is concave and nonnegative on } [0,T]_{\mathbb{T}}\}.$$

Suppose that there exists $l_1 \in \mathbb{T}$ such that $0 < l_1 < \xi_1 < T$ and $\int_0^{l_1} g(r)\nabla r > 0$ hold.

Lemma 3.3. *If $u \in P_1$, then*

(i) $u(t) \ge \frac{T-t}{T}\|u\|$ *for* $t \in [0,T]_{\mathbb{T}}$;

(ii) $(T-s)u(t) \ge (T-t)u(s)$ *for* $t, s \in [0,T]_{\mathbb{T}}$, *with* $s \le t$.

Proof. (i) Since $u^{\Delta\nabla}(t) \le 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Thus, for $0 < t < T$,

$$u(t) - u(0) = \int_0^t u^{\Delta}(s)\Delta s \ge t u^{\Delta}(t),$$
$$u(T) - u(t) = \int_t^T u^{\Delta}(s)\Delta s \le (T-t)u^{\Delta}(t)$$

from which we have

$$u(t) \ge \frac{tu(T) + (T-t)u(0)}{T} \ge \frac{T-t}{T}u(0) = \frac{T-t}{T}\|u\|.$$

(ii) If $t = s$, then the conclusion of (ii) holds. If $t > s$, $t, s \in [0,T]_{\mathbb{T}}$, setting $x(t) = u(t) - \frac{T-t}{T-s}u(s)$, for $u \in P_1$, we have

$$x^{\Delta\nabla}(t) = u^{\Delta\nabla}(t) \le 0, x(T) = u(T) \ge 0, x(s) = 0.$$

Therefore, the concavity of x implies that $x(t) \ge 0, t \in (s,T]_{\mathbb{T}}$, i.e., $(T-s)u(t) > (T-t)u(s)$, for $t > s$, $t, s \in [0,T]_{\mathbb{T}}$. This completes the proof. □

We define the nonnegative continuous concave functionals α_1, ψ_1 and the nonnegative continuous convex functionals $\gamma_1, \beta_1, \theta_1$ on the cone P_1 by

$$\gamma_1(u) = \theta_1(u) := \max_{t\in[\xi_1,T]_{\mathbb{T}}} u(t) = u(\xi_1),$$
$$\alpha_1(u) := \min_{t\in[0,l_1]_{\mathbb{T}}} u(t) = u(l_1),$$
$$\beta_1(u) := \max_{t\in[l_1,T]_{\mathbb{T}}} u(t) = u(l_1),$$
$$\psi_1(u) := \min_{t\in[0,\xi_1]_{\mathbb{T}}} u(t) = u(\xi_1).$$

We see that, for all $u \in P_1$, $\alpha_1(u) = u(l_1) = \beta_1(u)$.

$$
\begin{aligned}
M_1 &= \Big(\frac{\sum_{i=1}^{m-2} b_i(T-\xi_i)}{1-\sum_{i=1}^{m-2} b_i} + T - \xi_1\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big),\\
m_1 &= \Big(\frac{\sum_{i=1}^{m-2} b_i(T-\xi_i)}{1-\sum_{i=1}^{m-2} b_i} + T - l_1\Big)\phi_q\Big(\int_0^{l_1} g(r)\nabla r\Big),\\
\lambda_{l_1} &= \Big(\frac{\sum_{i=1}^{m-2} b_i(T-\xi_i)}{1-\sum_{i=1}^{m-2} b_i} + T - l_1\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big).
\end{aligned}
$$

We note that $u(t)$ is a solution of (1.1) and (1.2), if and only if

$$
\begin{aligned}
u(t) = {}& \frac{\sum_{i=1}^{m-2} b_i \left(\int_{\xi_i}^T \phi_q\big(\int_0^s g(r)f(u(r))\nabla r\big)\Delta s\right)}{1-\sum_{i=1}^{m-2} b_i}\\
&+ \int_t^T \phi_q\Big(\int_0^s g(r)f(u(r))\nabla r\Big)\Delta s, \quad t \in [0,T]_{\mathbb{T}}.
\end{aligned}
$$

Theorem 3.4. *Let $0 < a < \frac{(T-l_1)b}{T} < \frac{(T-l_1)(T-\xi_1)c}{T^2}$, $M_1 b < m_1 c$, and assume that f satisfies the following conditions:*

- (D1) $f(w) < \phi_p\big(\frac{c}{M_1}\big)$ *for all* $0 \le w \le \frac{Tc}{T-\xi_1}$*;*
- (D2) $f(w) > \phi_p\big(\frac{b}{m_1}\big)$ *for all* $b \le w \le \frac{T^2 b}{(T-\xi_1)^2}$*;*
- (D3) $f(w) < \phi_p\big(\frac{a}{\lambda_{l_1}}\big)$ *for all* $0 \le w \le \frac{Ta}{T-l_1}$.

Then, there exist at least three positive solutions u_1, u_2, u_3 of (1.1) and (1.3) such that

$$
\max_{t\in[l_1,T]_{\mathbb{T}}} u_1(t) < a, \quad b < \min_{t\in[0,l_1]_{\mathbb{T}}} u_2(t), \quad a < \max_{t\in[l_1,T]_{\mathbb{T}}} u_3(t) \quad \text{with} \ \min_{t\in[0,l_1]_{\mathbb{T}}} u_3(t) < b.
$$

Proof. Defining a completely continuous integral operator $F_1 : P_1 \to E$ by

$$
\begin{aligned}
(F_1u)(t) = {}& \frac{\sum_{i=1}^{m-2} b_i \left(\int_{\xi_i}^T \phi_q\left(\int_0^s g(r)f(u(r))\nabla r\right)\Delta s\right)}{1-\sum_{i=1}^{m-2} b_i}\\
&+ \int_t^T \phi_q\Big(\int_0^s g(r)f(u(r))\nabla r\Big)\Delta s, \quad u \in P_1,
\end{aligned} \tag{3.2}
$$

for $t \in [0,T]_{\mathbb{T}}$, each fixed point of F_1 in the cone P_1 is a positive solution of (1.1) and (1.3). We note that, if $u \in P_1$, then $(F_1u)(t) \ge 0$ for $t \in [0,T]_{\mathbb{T}}$, and

$$
(F_1u)^{\Delta}(t) = -\phi_q\Big(\int_0^t g(r)f(u(r))\nabla r\Big), \quad u \in P_1,\ t \in [0,T]_{\mathbb{T}^\kappa}.
$$

Note that $(F_1u)^{\Delta}(t)$ is continuous and nonincreasing on $[0,T]_{\mathbb{T}^\kappa}$, and $(F_1u)^{\Delta\nabla}(t) \le 0$ for $t \in [0,T]_{\mathbb{T}^\kappa\cap\mathbb{T}_\kappa}$. In addition, $(F_1u)^{\Delta}(0) = 0$. This implies $F_1u \in P_1$, and therefore $F : P_1 \to P_1$. In likeness to the proof of Theorem 3.2, we arrive at the conclusion. $\square$

4. An example

Let $\mathbb{T}=\{2-(\frac{1}{2})^{\mathbb{N}_0}\}\cup\{0,\frac{1}{8},\frac{1}{4},\frac{1}{3},\frac{1}{2},1,\frac{3}{2},2\}\cup[\frac{1}{10},\frac{1}{9}]$. We consider the p-Laplacian dynamic equation with $k\in\mathbb{N}_0$,

$$(\phi_p(u^\Delta(t)))^\nabla+\Big\{\sum_{k=0}^{6}t^k(\rho(t))^{6-k}\Big\}t^\nabla f(u(t))=0,\quad t\in[0,2]_{\mathbb{T}}, \tag{4.1}$$

satisfying the boundary conditions

$$u(0)=\frac{1}{2}u\Big(\frac{1}{4}\Big)+\frac{1}{6}u\Big(\frac{1}{2}\Big),\quad u^\Delta(2)=0, \tag{4.2}$$

where $p=4/3$, $\xi_1=1/4$, $\xi_2=1/2$, $a_1=1/2$, $a_2=1/6$, $T=2$ and

$$f(u)=\begin{cases}1\times 10^{-7}, & 0\le u\le 4,\\ p(u), & 4\le u\le 10,\\ 7\times 10^{-6}, & 10\le u\le 800,\\ s(u), & u\ge 800,\end{cases}$$

here $p(u)$ and $s(u)$ satisfy $p(4)=1\times 10^{-7}$, $p(10)=7\times 10^{-6}$, $s(800)=7\times 10^{-6}$, $(p^\nabla(u))^\nabla=0$ for $u\in(4,10)$, and $s(u):\mathbb{R}\to\mathbb{R}^+$ is continuous. If

$$g(t)=\Big\{\sum_{k=0}^{6}t^k(\rho(t))^{6-k}\Big\}t^\nabla,$$

then we obtain $(t^7)^\nabla=\Big\{\sum_{k=0}^{6}t^k(\rho(t))^{6-k}\Big\}t^\nabla$.

Choose $a=2$, $b=10$, $c=200$, $l=1$. Then

$$\begin{aligned}
M&=\Big(\frac{\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}a_i}+\xi_{m-2}\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big)\\
&=\big(\frac{5}{8}+\frac{1}{2}\big)\Big(\int_0^2\Big\{\sum_{k=0}^{6}t^k(\rho(t))^{6-k}\Big\}t^\nabla\nabla t\Big)^3\\
&=\frac{9}{8}\times 2^{21}=2.3593\times 10^6,\\
m&=\Big(\frac{\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}a_i}+l\Big)\phi_q\Big(\int_l^T g(r)\nabla r\Big)\\
&=\Big(\frac{5}{8}+1\Big)\Big(\int_1^2\Big\{\sum_{k=0}^{6}t^k(\rho(t))^{6-k}\Big\}t^\nabla\nabla t\Big)^3\\
&=\frac{13}{8}\times(2^7-1)^3=3.3286\times 10^6,\\
\lambda_l&=\Big(\frac{\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}a_i}+l\Big)\phi_q\Big(\int_0^T g(r)\nabla r\Big)\\
&=\frac{13}{8}\Big(\int_0^2\Big\{\sum_{k=0}^{6}t^k(\rho(t))^{6-k}\Big\}t^\nabla\nabla t\Big)^3\\
&=\frac{13}{8}\times 2^{21}=3.4079\times 10^6.
\end{aligned}$$

It is easy to see that
$$0 < a < \frac{lb}{T} < \frac{\xi_2 lc}{T^2}, \quad Mb < mc$$
and f satisfies
$$f(w) < \phi_p\left(\frac{a}{\lambda_l}\right) = 5.8687 \times 10^{-7}, \quad \text{for } 0 \le w \le \frac{Ta}{l} = 4,$$
$$f(w) > \phi_p\left(\frac{b}{m}\right) = 3.0043 \times 10^{-6}, \quad \text{for } 10 \le w \le \frac{T^2 b}{\xi_{m-2}^2} = 160,$$
$$f(w) < \phi_p\left(\frac{c}{M}\right) = 8.4771 \times 10^{-5}, \quad \text{for } 0 \le w \le \frac{Tc}{\xi_{m-2}} = 800.$$
So, all the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, the problem (4.1), (4.2) has at least three positive solutions u_1, u_2 and u_3 satisfying
$$\max_{t\in[0,1]_{\mathbb{T}}} u_1(t) < 2, \quad 10 < \min_{t\in[1,2]_{\mathbb{T}}} u_2(t), \quad 2 < \max_{t\in[0,1]_{\mathbb{T}}} u_3(t) \quad \text{with} \min_{t\in[1,2]_{\mathbb{T}}} u_3(t) < 10.$$

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