Haar wavelet collocation method for linear first order stiff differential equations

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Abstract. In general, there are countless types of problems encountered from different disciplines that can be represented by differential equations. These problems can be solved analytically in simpler cases; however, computational procedures are required for more complicated cases. Right at this point, the wavelet-based methods have been using to compute these kinds of equations in a more effective way. The Haar Wavelet is one of the appropriate methods that belongs to the wavelet family using to solve stiff ordinary differential equations (ODEs). In this study, The Haar Wavelet method is applied to stiff differential problems in order to demonstrate the accuracy and efficacy of this method by comparing the exact solutions. In comparison, similar to the exact solutions, the Haar wavelet method gives adequate results to stiff differential problems.

1 Introduction

Diverse types of engineering and practical problems are explained and interpreted as stiff ordinary differential equations (ODEs) [1]. While some problems can be solved by analytical methods, some cases are not always suitable for having an analytical solution. However, this factor itself is not the only reason to use numerical approaches. In practical use, even finding an analytical solution is not always helpful since the required computation power is exceedingly high[2].

Knowing only the differential equation is not sufficient to find the solutions to the problem, meaning that the need for extra information is evident. Since the proposed solution to the given stiff linear ODE is an initial value problem (IVP), the explanation of extra information can be said as "Initial Conditions". When all the components of y are defined in terms of x, which is specified at a certain value, then the problem can be classified as "Initial Value Problem". Therefore, an initial value problem is in the structure of;

$$y'(x) = f(y(x)), \text{ where } y(x_0) = y_0 \tag{1}$$

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Various types of other numerical methods have been used to solve these problems previously [3]. In this paper, the Haar Wavelet Method is used to solve such differential equations numerically. The proposed method was first used by Chen and Hsiao (1997) and was applied to the stiff systems of linear ODEs by Lepik (2009) [4–6]. The aim of this work is to discuss the quality, efficiency and accuracy of the Haar wavelet method for solving linear stiff ODEs with initial conditions. The resolution and accuracy of the method can be altered by changing only the value of J, which will be explained in the following sections. Typical computation time is taken for the solution of such problems, at $J = 7, 8, 9$ which yields to very low errors, just takes a few seconds in an average commercial computer with 2 core processors. When the required computation power and low error deviation is taken into consideration, the proposed solution shows the effectiveness of Haar Wavelet Method.

2 Haar Wavelets Collocation Methods

In 1910, a Hungarian mathematician who is Alfred Haar introduced the general form of Haar wavelets which are based on the functions. Haar wavelets are the simplest method in the wavelet families as mathematically and Haar wavelets consist of piecewise constant functions [5]. The following parameters are used in using this method: The function considered to be in the interval $[A, B]$, where A and B are constant numbers. The quantity M was defined as $M = 2^J$, where J defines the maximal level of resolution and $\Delta x = \frac{B-A}{2M}$, where Δx is the length of each equal subinterval [5, 6]. Other two parameters can be identified as

$$j = 0, 1, \ldots ..J$$
$$k = 0, 1, \ldots \ldots ., m - 1$$

in which j is the dilatation parameter and k is the translation parameter. m, which is introduced in the parameter k, is $m = 2^j$. The wavelet numbers are denoted by the letter i, which is identified as [6, 7]:

$$i = m + k + 1$$

The following formula represents the i-th Haar wavelet [5].

$$h_i(x) = \begin{cases} 1 \text{ for } x \in [\xi_1(i), \xi_2(i)) \\ -1 \text{ for } x \in [\xi_2(i), \xi_3(i)) \\ 0 \text{ elsewhere} \end{cases}, \tag{2}$$

where

$$\xi_1(i) = A + 2k\mu\Delta x, \qquad \xi_2(i) = A + (2k+1)\mu\Delta x$$
$$\xi_3(i) = A + 2(k+1)\mu\Delta x, \quad \mu = \frac{M}{m}$$

With the help of the parameters described above the Haar function can be determined. Haar functions are used to define the Haar matrix which consists of only 1,-1,and 0. To solve the differential equation another operational matrix, which is P, are needed.

$$\int_A^B h_i(x) h_l(x) dx = \begin{cases} (B-A)2^{-j} & \text{for } l = i \\ 0 & \text{for } l \neq i \end{cases} \tag{3}$$

The integrals of the Haar functions are required below.

$$p_{\alpha,i}(x) = \begin{cases} 0 & \text{for } x < \xi_1(i) \\ \frac{1}{\alpha!}\left[x - \xi_1(i)\right]^\alpha & \text{for } x \in [\xi_1(i), \xi_2(i)] \\ \frac{1}{\alpha!}\left\{\left[x - \xi_1(i)\right]^\alpha - 2\left[x - \xi_2(i)\right]^\alpha\right\} & \text{for } x \in [\xi_2(i), \xi_3(i)] \\ \frac{1}{\alpha!}\left\{\left[x - \xi_1(i)\right]^\alpha - 2\left[x - \xi_2(i)\right]^\alpha + \left[x - \xi_3(i)\right]^\alpha\right\} & \text{for } x > \xi_3(i) \end{cases} \tag{4}$$

Both Haar and operational matrices are constructed in a same way that Lepik (2009) explained in his work [6]. Moreover, to find the Haar function and p function, we need to use collocation points. In other words, Haar matrices and P matrices can be calculated with the help of collocation points [5–7].

3 STIFF ODEs

As it is previously introduced, stiff ordinary differential equations have been using in many different fields such as in fluid mechanics, elasticity, electrical networks, chemical reactions, etc.[4, 8]

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b \tag{5}$$

Eqn. 5 is the general form of the first-order linear system of ODEs where f(x,y) can be defined in the interval [a,b]. Stiffness of Eqn. 5 can be checked as[9]; if the following expressions are valid, then a linear system is stiff,

- $\mathrm{Re}\,(\lambda_s(x)) < 0, \mathrm{s} = 1, 2, 3, \ldots, \mathrm{m}$

- $\max|\mathrm{Re}\,(\lambda_s)| \gg \min|\mathrm{Re}\,(\lambda_s)|\ \mathrm{s} = 1, 2, 3, \ldots, \mathrm{m}$

The ratio of $\max|\mathrm{Re}\,(\lambda_s)|$ and $\min|\mathrm{Re}\,(\lambda_s)|$ is called the stiffness ratio and λ_s, which is expressed in the first and second cases are the eigenvalues of the Jacobian of the system [10].

4 APPLICATIONS

In this section, several stiff differential problems solved by Haar Wavelet method, which is investigated in previous sections.

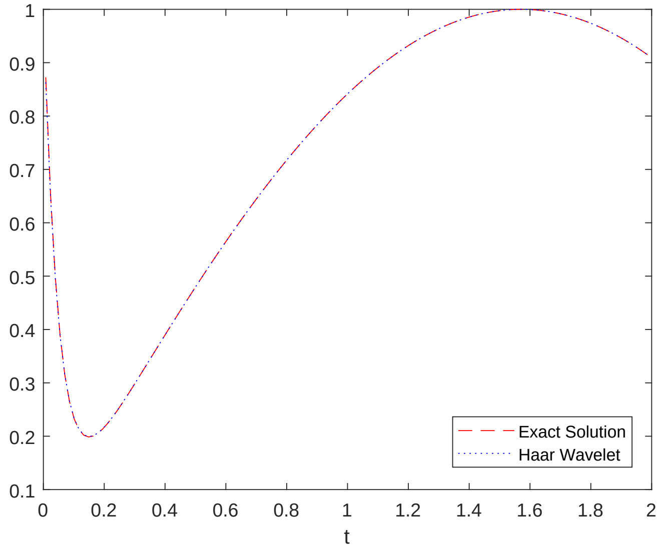

Figure 1. Exact solution and the approximate solutions calculated for $J = 6$ and $i = 128$ by Haar Wavelet for Eqn. 6

4.1 Example 1

The first example of the first order stiff differential problem is formulated as follows[11]:

$$y' = -20y + 20\sin t + \cos t, \quad 0 \leq t \leq 2, \text{ with } y(0) = 1 \tag{6}$$

And the exact solution is given as:

$$y(t) = \sin t + e^{-20t} \tag{7}$$

Fig. 1 indicates the exact solution and Haar Wavelet solution. The resolution number J is designated as 6 for this solution. Fig. 1 demonstrated that solution computed by Haar Wavelet method converge to the exact solution.

Table 1. Numerical results, exact solutions and errors for Example 1 for J=8

t	Haar Solution	Exact Solution	Absolute Error	Logarithmic Error
0.002	0.964359136	0.963643725	7.15E − 04	-3.1457
0.2	0.217686034	0.217709067	2.30E − 05	-4.6383
0.4	0.390109793	0.390110954	1.16E − 06	-5.9355
0.6	0.564326139	0.564326226	8.71E − 08	-7.0600
0.8	0.719258524	0.719258571	4.71E − 08	-7.3270
1	0.842524622	0.842524659	3.69E − 08	-7.4330
1.2	0.932463058	0.932463084	2.59E − 08	-7.5867
1.4	0.985516034	0.985516048	1.39E − 08	-7.8570
1.6	0.999584932	0.999584933	1.34E − 09	-8.8729
1.8	0.973222747	0.973222735	1.15E − 08	-7.9393
2	0.910108502	0.910108479	2.34E − 08	-7.6308

Table 1 represents the numerical solutions and exact solution at selected values of time for $J = 8$. Absolute error or the numerical solutions are also represented in the tables at the last column. To be able to compare the effect of J, one can use Table 1 and Table 2 which represents solutions for $J = 12$.

Table 2. Numerical results , exact solutions and errors for Example 1 for J=12

t	Haar Solution	Exact Solution	Absolute Error	Logarithmic Error
0.00012	0.997686610	0.997683642	2.97E − 06	-5.5275
0.2	0.217029850	0.217029941	9.11E − 08	-7.0403
0.4	0.389776123	0.389776128	4.56E − 09	-8.3407
0.6	0.564628470	0.564628470	3.40E − 10	-9.4689
0.8	0.717305173	0.717305173	1.85E − 10	-9.7334
1	0.841536935	0.841536935	1.44E − 10	-9.8406
1.2	0.932065623	0.932065623	1.01E − 10	-9.9943
1.4	0.985453879	0.985453879	5.43E − 11	-10.2651
1.6	0.999574316	0.999574316	5.15E − 12	-11.2879
1.8	0.973864269	0.973864269	4.42E − 11	-10.3545
2	0.909348219	0.909348219	9.18E − 11	-10.0371

4.2 Example 2

The second example of the first order stiff differential problem is formulated as follows[12]:

$$y' + 250y = 250 \sin t + \cos t, \quad 0 \leq t \leq 1, \text{ with } y(0) = 1 \tag{8}$$

The Exact Solution is given as;

$$y(t) = e^{-250t} + \sin t \tag{9}$$

Fig. 2 provides the solutions which belong to Exact solution and Haar Wavelet method respectively. This particular solution computed by Haar Wavelet method for $J = 6$. The results of the Haar wavelet method and results of exact solution are matching precisely.

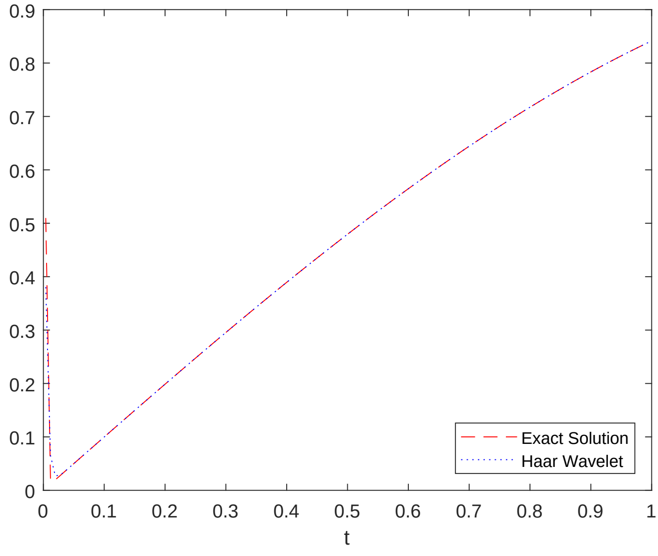

Figure 2. Exact solution and the approximate solutions calculated for $J = 6$ and $i = 128$ by Haar Wavelet for Eqn. 8

Table 3. Numerical results, exact solutions and errors for Example 2 for J=6

t	Haar Solution	Exact Solution	Absolute Error	Logarithmic Error
0.0039	0.509835084	0.380509691	0.1293	-0.8883
0.1	0.097501083	0.097501104	2.03E – 08	-7.6929
0.2	0.197903573	0.197903593	2.00E – 08	-7.6998
0.3	0.296266454	0.296266473	1.95E – 08	-7.7109
0.4	0.391575989	0.391576007	1.88E – 08	-7.7269
0.5	0.482849911	0.482849929	1.79E – 08	-7.7482
0.6	0.562706527	0.562706544	1.69E – 08	-7.7730
0.7	0.643619942	0.643619958	1.56E – 08	-7.8062
0.8	0.71790016	0.717900174	1.42E – 08	-7.8470
0.9	0.784781643	0.784781655	1.28E – 08	-7.8971
1	0.839354003	0.839354014	1.11E – 08	-7.9536

Table 3 represents the numerical solutions and exact solution at selected values of x for $J = 6$. Absolute error or the numerical solutions are also represented in the tables at the last column. To be able to compare the effect of J, one can use Table 3 and Table 4 which represents solutions for $J = 10$.

Table 4. Numerical results and exact solutions for Example 2 for J=10

t	Haar Solution	Exact Solution	Absolute Error	Logarithmic Error
0.00024	0.942719981	0.941034305	0.0017	-2.7732
0.1	0.099687663	0.099687663	7.95E − 11	-10.099
0.2	0.198621476	0.198621476	7.80E − 11	-10.107
0.3	0.295566854	0.295566854	7.60E − 11	-10.1189
0.4	0.389553259	0.389553259	7.34E − 11	-10.1341
0.5	0.479639778	0.479639778	7.00E − 11	-10.1550
0.6	0.564521568	0.564521569	6.61E − 11	-10.1797
0.7	0.644180341	0.644180341	6.10E − 11	-10.2145
0.8	0.717390109	0.717390109	5.56E − 11	-10.2550
0.9	0.783417957	0.783417957	5.00E − 11	-10.3014
1	0.84133905	0.84133905	4.30E − 11	-10.3662

5 CONCLUSION

The Haar wavelet has a wide range of usage areas such as linear and nonlinear ODE as denoted at earlier of the study. The investigation indicates the applicability of the Haar Wavelet which belongs to the Wavelet family to the several types of linear stiff differential initial problems. The results of example 1 and 2 are computed by the Haar wavelet method compared with their exact solution results. Consequently, the numerical values and graphs show that this approach is efficient and the results can be obtained rapidly with the aid of the construction of the Haar matrices.

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