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Multiple positive solutions of nonlinear m-point dynamic equations for p-Laplacian on time scales

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Abstract: In this paper, we study the existence of positive solutions of a nonlinear m-point p-Laplacian dynamic equation

$$(\phi_p(x^{\Delta}(t)))^{\nabla} + w(t)f(t, x(t), x^{\Delta}(t)) = 0, \qquad t_1 < t < t_m,$$

subject to one of the following boundary conditions

$$x(t_1) - B_0\Big(\sum_{i=2}^{m-1} a_i x^{\Delta}(t_i)\Big) = 0, \qquad x^{\Delta}(t_m) = 0,$$

or

$$x^{\Delta}(t_1) = 0, \qquad x(t_m) + B_1\Big(\sum_{i=2}^{m-1} b_i x^{\Delta}(t_i)\Big) = 0,$$

where $\phi_p(s) =| s |^{p-2} s, \quad p > 1$. Sufficient conditions for the existence of at least three positive solutions of the problem are obtained by using a fixed point theorem. The interesting point is the nonlinear term f is involved with the first order derivative explicitly. As an application, an example is given to illustrate the result.

Key words: Time scales, boundary value problem, p-Laplacian, positive solutions, fixed point theorem

1. Introduction

The theory of dynamic equations on time scales was introduced by Stefan Hilger [14] in his PhD thesis in 1988. It has been created in order to unify continuous and discrete analysis, and it allows simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. Moreover, the study of time scales has led to a number of significant applications, e.g., in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing, and epidemic models.

In [18], Su and Li studied the existence of positive solutions of p-Laplacian dynamic equation

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + a_1(t)f(u(t)) = 0 \qquad t \in [0, T]_{\mathbb{T}},$$

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subject to boundary conditions

$$u(0) - B_0\Big(\sum_{i=1}^{m-2} a_i u^{\Delta}(\xi_i)\Big) = 0, \qquad u^{\Delta}(T) = 0,$$

or

$$u^{\Delta}(0) = 0, \qquad x(T) + B_1\Big(\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i)\Big) = 0,$$

where $\varphi_p(v) = |v|^{p-2}v$ with $p > 1$. They showed that the boundary value problem has at least three positive solutions by using the five functional fixed-point theorem.

In [19], Sun and Li studied the following p-Laplacian m- point boundary value problem on time scales

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t, u(t)) = 0, \qquad t \in (0, T),$$

$$u(0) = 0, \qquad \varphi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} a_i \varphi_p(u^{\Delta}(\xi_i)),$$

where $a \in C_ld\ ((0,T),[0,\infty))$ and $f \in C_ld\ ((0,T) \times [0,\infty),[0,\infty))$. They found some new results for the existence of at least twin or triple positive solutions of the problem by applying Avery–Henderson and Leggett–Williams fixed point theorems, respectively.

In [20] the authors considered the following p-Laplacian multipoint boundary value problem on time scales:

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t, u(t)) = 0, \qquad t \in [0, T]_{\mathbb{T}},$$

$$\phi_p(u^{\Delta}(0)) = \sum_{i=1}^{n-2} a_i \phi_p(u^{\Delta}(\xi_i)), \qquad u(T) = \sum_{i=1}^{n-2} b_i \phi_p u(\xi_i),$$

where $\phi_p(s) = |s|^{p-2}s$ with $p > 1, \xi_i \in [0,T]_{\mathbb{T}}, 0 < \xi_1 < \xi_2 < \ldots < \xi_{n-2} < \rho(T)$. They provided some sufficient conditions for the existence of multiple positive solutions to the problem by using a fixed point index.

Recently, much attention has been paid to the existence of positive solutions of boundary value problems (BVPs) on time scales; see [1, 6, 9, 10, 13, 15–20]. However, to the best of our knowledge, there are not many results concerning p-Laplacian dynamic equations with nonlinearity depending on the first order derivative for BVPs on time scales [7, 8, 11].

Motivated by the above works, in this paper, we consider the existence of at least three positive solutions for a p-Laplacian dynamic equation on time scales,

$$(\phi_p(x^{\Delta}(t)))^{\nabla} + w(t)f(t, x(t), x^{\Delta}(t)) = 0, \qquad t_1 < t < t_m, \tag{1.1}$$

subject to one of the following boundary conditions:

$$x(t_1) - B_0\Big(\sum_{i=2}^{m-1} a_i x^{\Delta}(t_i)\Big) = 0, \qquad x^{\Delta}(t_m) = 0, \tag{1.2}$$

or

$$x^{\Delta}(t_1) = 0, \qquad x(t_m) + B_1\Big(\sum_{i=2}^{m-1} b_i x^{\Delta}(t_i)\Big) = 0, \tag{1.3}$$

where $\phi_p(u)$ is a p-Laplacian operator, i.e. $\phi_p(s) =| s |^{p-2} s$, for $p > 1$, with $(\phi_p)^{-1} = \phi_q$ and $\frac{1}{p}+\frac{1}{q} = 1$, and the points $t_i \in \mathbb{T}_k^k$ for $i \in \{1, 2, \ldots, m\}$ with $0 = t_1 < t_2 < \ldots < t_m = 1$. The usual notation and terminology for time scales as can be found in [3, 4] will be used here. The interesting point is that the nonlinear term f is associated with the first order derivative explicitly and the main tool is a fixed point theorem due to Avery and Peterson. The results are even new for the special cases of difference equations and differential equations, as well as in the general time scale setting.

Throughout the paper, we will suppose that the following conditions are satisfied:

(H1) $a_i, b_i \in [0, \infty)$, $i \in \{2, 3, \ldots, m-1\}$ with $0 < \sum_{i=2}^{m-1} a_i < 1$, and $\sum_{i=2}^{m-1} b_i < 1$

(H2) $w(t) \in C_{ld}([t_1, t_m], [0, +\infty))$ and does not vanish identically on any closed subinterval of $[t_1, t_m]$, where $C_{ld}([t_1, t_m], [0, +\infty))$ denotes the set of left dense continuous from $\mathbb{T}$ to $[0, +\infty)$

(H3) $f : [t_1, t_m] \times [0, +\infty) \times \mathbb{R} \longrightarrow [0, +\infty)$ is continuous;

(H4) B_0 and B_1 satisfy $Bv \le B_i(v) \le Av$, $v \in \mathbb{R}$, $i = 0, 1$, here B and A are nonnegative numbers.

2. Preliminaries

In this section, we provide some background materials from theory of cones in Banach spaces. The following definitions can be found in the book by Deimling [5] as well as in the book by Guo and Lakshmikantham [12].

Definition 2.1 Let E be a real Banach space over $\mathbb{R}$. A nonempty closed set $P \subset E$ is said to be a cone if it satisfies the following two conditions:

(i) $au + bv \in P$ for all $u, v \in P$ and all $a, b \ge 0$;

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \le y$ if and only if $y - x \in P$.

Definition 2.2 An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3 A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha :\longrightarrow [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Similarly, we say the map γ is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\gamma : P \longrightarrow [0, \infty)$ is continuous and

$$\gamma(tx + (1-t)y) \le t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following sets:

$$\begin{aligned}
P(\gamma, d) &= \{x \in P : \gamma(x) < d\}, \\
P(\gamma, \alpha, b, d) &= \{x \in P : b \leq \alpha(x), \gamma(x) \leq d\}, \\
P(\gamma, \theta, \alpha, b, c, d) &= \{x \in P : b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\}, \\
R(\gamma, \psi, a, d) &= \{x \in P : a \leq \psi(x), \gamma(x) \leq d\}.
\end{aligned}$$

To prove our results, we need the following fixed point theorem due to Avery and Peterson.

Theorem 2.4 *([2, Theorem 10]). Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda \psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,*

$$\alpha(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x) \tag{2.1}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(i) $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq \emptyset, \quad$ *and* $\quad \alpha(Tx) > b \quad$ *for* $\quad x \in P(\gamma, \theta, \alpha, b, c, d)$;

(ii) $\alpha(Tx) > b, \quad$ *for* $\quad x \in P(\gamma, \alpha, b, d) \quad$ *with* $\quad \theta(Tx) > c$;

(iii) $0 \notin R(\gamma, \psi, a, d) \quad$ *and* $\quad \psi(Tx) < a \quad$ *for* $\quad x \in R(\gamma, \psi, a, d) \quad$ *with* $\quad \psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d \quad \text{for} \quad i = 1, 2, 3, \quad b < \alpha(x_1), \quad a < \psi(x_2), \quad \text{with} \;\; \alpha(x_2) < b, \quad \psi(x_3) < a.$$

3. Existence of multiple positive solutions to (1.1) and (1.2)

In this section, we shall obtain existence results for the problems (1.1) and (1.2) by using the Avery–Peterson fixed point theorem.

We define the real Banach space $E = C^{\Delta}[t_1, \sigma(t_m)]$ to be the set of all delta-differential functions with continuous delta-derivative on $[t_1, \sigma(t_m)]$ with the norm

$$\|x\|_{1,\mathbb{T}} = \max\{\|x\|_{0,\mathbb{T}}, \|x^{\Delta}\|_{0,\mathbb{T}^k}\}, \quad x \in E,$$

where

$$\|x\|_{0,\mathbb{T}} := \sup\{|x(t)| : t \in [t_1, t_m]\}, \quad \|x^{\Delta}\|_{0,\mathbb{T}^k} := \sup\{|x^{\Delta}(t)| : t \in [t_1, t_m]_{\mathbb{T}^k}\}, \quad x \in E.$$

From the fact

$$(\phi_p(x^{\Delta}(t)))^{\nabla} = -w(t)f(t, x(t), x^{\Delta}(t)) \leq 0 \;\; \text{for} \;\; t \in [t_1, t_m]_{\mathbb{T}^k_k},$$

we know that x is concave on $[t_1, t_m]$. Therefore, define a cone $P_1 \subset E$ by

$$P_1 = \left\{ x \in E : x(t) \geq 0, x(t_1) - B_0 \left(\sum_{i=2}^{m-1} a_i x^{\Delta}(t_i) \right) = 0, x \text{ is concave on } [t_1, t_m] \right\} \subset E.$$

Let the nonnegative continuous concave functional α_1, the nonnegative continuous convex functional θ_1, γ_1, and the nonnegative continuous functional ψ_1 be defined on the cone P_1 by

$$\begin{aligned} \alpha_1(x) &= \min_{t \in [\frac{1}{n}, \frac{n-1}{n}]_{\mathbb{T}}} |x(t)|, \qquad \gamma_1(x) = \max_{t \in [t_1, t_m]_{\mathbb{T}^k}} | x^{\Delta}(t) |, \\ \psi_1(x) &= \theta_1(x) = \max_{t \in [t_1, t_m]} | x(t) |, \qquad x \in P_1, \end{aligned}$$

where $0 = t_1 < \frac{1}{n} < t_2 < \ldots < t_{m-1} = t_m - \frac{2}{n} < t_m = 1, \quad n > \max \left\{ \frac{1}{t_2}, \frac{2}{t_m - t_{m-1}} \right\}$.

Lemma 3.1 *Let* $0 = t_1 < \frac{1}{n} < t_2 < \ldots < t_{m-1} = t_m - \frac{2}{n} < t_m = 1$*. If (H1) holds, then for* $x \in P_1$*, there exists a constant* $M > 0$ *such that*

$$\max_{t \in [t_1, t_m]} | x(t) | \leq M \max_{t \in [t_1, t_m]_{\mathbb{T}^k}} | x^{\Delta}(t) |,$$

where

$$M = A \sum_{i=2}^{m-1} a_i + t_m - t_1.$$

Proof Since

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(s) \Delta s,$$

so we have

$$\begin{aligned} \max_{t \in [t_1, t_m]} | x(t) | &\leq | x(t_1) | + (t_m - t_1) \max_{t \in [t_1, t_m]_{\mathbb{T}^k}} | x^{\Delta}(t) | \\ &= \left| B_0 \sum_{i=2}^{m-1} a_i x^{\Delta}(t_i) \right| + (t_m - t_1) \max_{t \in [t_1, t_m]_{\mathbb{T}^k}} | x^{\Delta}(t) | \\ &\leq \left(A \sum_{i=2}^{m-1} a_i + t_m - t_1 \right) \max_{t \in [t_1, t_m]_{\mathbb{T}^k}} | x^{\Delta}(t) | . \end{aligned}$$

□

By Lemma 3.1 and the concavity of x, for all $x \in P_1$, the functionals defined above hold for the relations

$$\frac{1}{n} \theta_1(x) \leq \alpha_1(x) \leq \theta_1(x), \qquad \|x\|_{1,\mathbb{T}} = \max\{\theta_1(x), \gamma_1(x)\} \leq M \gamma_1(x). \tag{3.1}$$

Thus, the condition (2.1) of Theorem 2.4 is satisfied.

Lemma 3.2 *If* $y \in C_{ld}[t_1, t_m]$, *then*

$$(\phi_p(x^{\Delta}(t)))^{\nabla} + y(t) = 0, \qquad t_1 < t < t_m, \tag{3.2}$$

$$x(t_1) - B_0\Big(\sum_{i=2}^{m-1} a_i x^{\Delta}(t_i)\Big) = 0, \qquad x^{\Delta}(t_m) = 0, \tag{3.3}$$

has a unique solution

$$x(t) = B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right)\right) + \int_{t_1}^{t} \phi_q\left(\int_{s}^{t_m} y(r)\nabla r\right)\Delta s. \tag{3.4}$$

Proof From (3.2), we have

$$(\phi_p(x^{\Delta}(t)))^{\nabla} = -y(t)$$

Integrating from t to t_m, we get

$$\phi_p(x^{\Delta}(t)) = \int_{t}^{t_m} y(r)\nabla r \qquad \text{i.e.,} \qquad x^{\Delta}(t) = \phi_q\left(\int_{t}^{t_m} y(r)\nabla r\right). \tag{3.5}$$

Integration from t_1 to t yields

$$x(t) - x(t_1) = \int_{t_1}^{t} \phi_q\left(\int_{s}^{t_m} y(r)\nabla r\right)\Delta s;$$

i.e.

$$x(t) = x(t_1) + \int_{t_1}^{t} \phi_q\left(\int_{s}^{t_m} y(r)\nabla r\right)\Delta s.$$

From (3.5), we get

$$x^{\Delta}(t_i) = \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right).$$

From (3.3), we have

$$x(t_1) - B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right)\right) = 0.$$

Thus,

$$x(t_1) = B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right)\right).$$

Therefore, (3.2) and (3.3) have a unique solution

$$x(t) = B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right)\right) + \int_{t_1}^{t} \phi_q\left(\int_{s}^{t_m} y(r)\nabla r\right)\Delta s.$$

□

Lemma 3.3 *Assume Lemma 3.2 holds. If* $y \in C_{ld}[t_1, t_m]$ *and* $y \geq 0$, *then the solution of BVPs (3.2) and (3.3) satisfies* $x(t) \geq 0$.

Proof By Lemma 3.2, we find

$$\begin{aligned} x(t_1) &= B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right)\right) \geq 0, \\ x(t_m) &= B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right)\right) \\ &\quad + \int_{t_1}^{t_m} \phi_q\left(\int_{s}^{t_m} y(r)\nabla r\right)\Delta s \geq 0. \end{aligned}$$

If $t \in (t_1, t_m)$, we have

$$\begin{aligned} x(t) &= B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} y(r)\nabla r\right)\right) \\ &\quad + \int_{t_1}^{t} \phi_q\left(\int_{s}^{t_m} y(r)\nabla r\right)\Delta s \geq 0. \end{aligned}$$

Therefore, $x(t) \geq 0$ for $t \in [t_1, t_m]$. □

From Lemma 3.2, it is easy to see that BVPs (1.1) and (1.2) have a solution $x = x(t)$ if and only if x solves the equation

$$\begin{aligned} x(t) &= B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\right)\right) \\ &\quad + \int_{t_1}^{t} \phi_q\left(\int_{s}^{t_m} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\right)\Delta s. \end{aligned}$$

We define the operator $F : P_1 \to P_1$ as follows

$$\begin{aligned} (Fx)(t) &:= B_0\left(\sum_{i=2}^{m-1} a_i \phi_q\left(\int_{t_i}^{t_m} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\right)\right) \\ &\quad + \int_{t_1}^{t} \phi_q\left(\int_{s}^{t_m} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\right)\Delta s. \end{aligned} \tag{3.6}$$

Taking the delta derivative of $(Fx)(t)$, we have

$$(Fx)^{\Delta}(t) := \phi_q\left(\int_{t}^{t_m} w(r)f(r, x(r), x^{\Delta}(r))\ \nabla r\right), \quad \text{for } x \in P_1, \quad t \in [t_1, t_m]_{\mathbb{T}^k}. \tag{3.7}$$

Lemma 3.4 *Let (H1)–(H4) hold. If conditions 1–3 are satisfied, then* $FP_1 \subset P_1$ *and* $F : P_1 \to P_1$ *is completely continuous.*

Proof

For $x \in P_1$, from the definition of the operator F, we deduce that there is $Fx \in C^{\Delta}[t_1, \sigma(t_m)]$, which is nonnegative and $(Fx)(t_1) - B_0\Big(\sum_{i=2}^{m-1} a_i x^{\Delta}(t_i)\Big) = 0$.

On the other hand, we have

$$(\phi_p(x^{\Delta}(t)))^{\nabla} = -w(t)f(t, x(t), x^{\Delta}(t)) \leq 0 \quad \text{for} \quad t \in [t_1, t_m]_{\mathbb{T}_k^k},$$

which implies that Fx is concave on $[t_1, t_m]$. Therefore, $F(P_1) \subset P_1$.

Next we shall prove that operator F is completely continuous.

(1) Operator F is continuous. Because the function f is continuous, this conclusion can be easily found.

(2) For each constant $l > 0$, let $B_l = \{x \in P_1 : \|x\|_{1,\mathbb{T}} \leq l\}$. Then B_l is a bounded closed convex set in P_1. $\forall x \in B_l$, from (3.6) and (3.7), we have

$$\begin{aligned} \|(Fx)(t)\|_{0,\mathbb{T}} &\leq A\sum_{i=2}^{m-1} a_i(\mathbb{MN})^{q-1} + (\mathbb{MD})^{q-1}(t - t_1) \\ &\leq A\sum_{i=2}^{m-1} a_i(\mathbb{MN})^{q-1} + (\mathbb{MD})^{q-1} \end{aligned}$$

and

$$\|(Fx)^{\Delta}(t)\|_{0,\mathbb{T}^k} \leq (\mathbb{MC})^{q-1}$$

where

$$\begin{aligned} \mathbb{M} &= \sup_{r\in[s,t_m]_{\mathbb{T}}, \|x\|_{1,\mathbb{T}}\leq l} f(t, x(t), x^{\Delta}(t)), \qquad \mathbb{N} = \int_{t_i}^{t_m} w(r)\nabla r, \\ \mathbb{C} &= \int_{t}^{t_m} w(r)\nabla r, \qquad \mathbb{D} = \int_{s}^{t_m} w(r)\nabla r, \qquad s \in [t_1, t_m], \qquad t \in [t_1, t_m]_{\mathbb{T}^k}. \end{aligned}$$

Therefore, $F(B_l)$ is uniformly bounded.

(3) The family $\{F_x : x \in B_l\}$ is a family of equicontinuous functions. Let $\bar{t}, t^{\star} \in [t_1, t_m]_{\mathbb{T}^k}$, $\bar{t} < t^{\star}$ and $B_l = \{x \in P_1 : \|x\|_{1,\mathbb{T}} \leq l\}$ be a bounded set of P_1.

Hence

$$\begin{aligned}
|(Fx)(\bar{t})-(Fx)(t^\star)| &= \left|\int_{t_1}^{\bar{t}}\phi_q\left(\int_s^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\Delta s\right.\\
&\quad\left.-\int_{t_1}^{t^\star}\phi_q\left(\int_s^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\Delta s\right|\\
&= \left|\int_{t_1}^{\bar{t}}\phi_q\left(\int_s^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\Delta s\right.\\
&\quad -\int_{t_1}^{\bar{t}}\phi_q\left(\int_s^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\Delta s\\
&\quad\left.-\int_{\bar{t}}^{t^\star}\phi_q\left(\int_s^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\Delta s\right|.
\end{aligned}$$

As $\bar{t}\to t^\star$, the right-hand side of the above inequality is independent of $y\in B_l$ and tends to zero. Similarly, we get

$$\begin{aligned}
|(Fx)^\Delta(\bar{t})-(Fx)^\Delta(t^\star)| &= \left|\phi_q\left(\int_{\bar{t}}^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\right.\\
&\quad\left.-\phi_q\left(\int_{t^\star}^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\right|\\
&= \left|\phi_q\left(\int_{\bar{t}}^{t^\star} w(r)f(r,x(r),x^\Delta(r))\nabla r\right.\right.\\
&\quad\left.+\int_{t^\star}^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\\
&\quad\left.-\phi_q\left(\int_{t^\star}^{t_m} w(r)f(r,x(r),x^\Delta(r))\nabla r\right)\right|\to 0,
\end{aligned}$$

as $\bar{t}\to t^\star$. Thus, the set $\{F_x : x\in B_l\}$ is equicontinuous.

As a consequence of (1)–(3) together with the Ascoli–Arzela theorem we can prove $F : P_1 \to P_1$ is completely continuous. □

Moreover, we can prove the following result:

$$\min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}}(Fx)(t) \geq \frac{1}{n}\|Fx\|_{0,\mathbb{T}} = \frac{1}{n}(Fx)(t_m). \tag{3.8}$$

In fact, the concavity of (Fx) on $[t_1,t_m]$, $t_m=1$ and (3.6) imply

$$\frac{(Fx)(t)}{t} \geq \frac{(Fx)(t_m)}{t_m} = \|Fx\|_{0,\mathbb{T}}, \quad \text{for } t\in\left[\frac{1}{n},\frac{n-1}{n}\right]_{\mathbb{T}},$$

which implies that (3.8) holds.

Remark: For convenience, we introduce the following notations. Let

$$\eta=\phi_q\left(\int_{t_1}^{t_m} w(r)\nabla r\right),\qquad \lambda_i=B\left(\sum_{i=2}^{m-1}a_i\phi_q\left(\int_{t_i}^{t_i^\star} w(r)\nabla r\right)\right),$$

$$\lambda=\min_{i\in[t_1,t_{m-1}]}\{\lambda_i\},\qquad t_i^\star=\frac{t_i+t_{i+1}}{2}\quad (i=1,2,\dots,m-1),$$

$$S_i=A\left(\sum_{i=2}^{m-1}a_i\phi_q\left(\int_{t_i}^{t_m} w(r)\nabla r\right)\right)+\int_{t_1}^{t_m}\phi_q\left(\int_{s}^{t_m} w(r)\nabla r\right)\Delta s,\quad S=\max_{i\in[t_1,t_{m-1}]}\{S_i\},$$

$$L=\frac{n}{2}\left[1+\left(1-\sum_{i=2}^{m-1}a_i\right)\right],\quad 0=t_1<\frac{1}{n}<t_2<\ldots<t_{m-1}=t_m-\frac{2}{n}<t_m=1.$$

Now we state and prove our main result.

Theorem 3.5 *Suppose that (H1)–(H4) hold. Let* $0<a<b\le\frac{2Md}{L}$, *and assume that* f *satisfies the following conditions:*

(H5) $f(t,u,v)\le\phi_p(d/\eta)$, *for* $(t,u,v)\in[t_1,t_m]\times[0,Md]\times[-d,d]$;

(H6) $f(t,u,v)>\phi_p(nb/\lambda)$, *for* $(t,u,v)\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}\times[b,Lb]\times[-d,d]$;

(H7) $f(t,u,v)<\phi_p(a/S)$, *for* $(t,u,v)\in[t_1,t_m]\times[0,a]\times[-d,d]$.

Theorem 2.4 holds. Then the BVPs (1.1) and (1.2) have at least three positive solutions x_1,x_2, *and* x_3 *such that*

$$\max_{t\in[t_1,t_m]_{\mathbb{T}^k}}\mid x_i^\Delta(t)\mid\le d,\qquad i=1,2,3;$$

$$b<\min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}}\mid x_1(t)\mid,\qquad \max_{t\in[t_1,t_m]}\mid x_1(t)\mid\le Md,$$

$$a<\max_{t\in[t_1,t_m]}\mid x_2(t)\mid,\quad \textit{with}\quad \min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}}\mid x_2(t)\mid<b,$$

$$\max_{t\in[t_1,t_m]}\mid x_3(t)\mid<a.$$

Proof We set out to verify that operator F satisfies the Avery–Peterson fixed point theorem, which will prove the existence of three fixed points of F that satisfy the conclusion of the theorem.

For $x\in\overline{P_1(\gamma_1,d)}$, there is $\gamma_1=\max_{t\in[t_1,t_m]}\mid x^\Delta(t)\mid\le d$. For $t\in[t_1,t_m]_{\mathbb{T}^k}$, by Lemma 3.1, there is $\max_{t\in[t_1,t_m]}\mid x(t)\mid\le Md$; then the condition (H5) implies that $f(t,x(t),x^\Delta(t))\le\phi_p(d/\eta)$. On the other hand, for $x \in P_1$, there is $Fx \in P_1$; then Fx is concave on $[t_1, t_m]$ and $\max_{t\in[t_1,t_m]_{\mathbb{T}^k}} \mid (Fx)^{\Delta}(t) \mid = (Fx)^{\Delta}(t_1)$. Therefore,

$$\begin{aligned}
\gamma_1(Fx) &= \max_{t\in[t_1,t_m]_{\mathbb{T}^k}} \mid (Fx)^{\Delta}(t) \mid = (Fx)^{\Delta}(t_1) \\
&= \phi_q\bigg(\int_{t_1}^{t_m} w(r) f(r, x(r), x^{\Delta}(r))\nabla r\bigg) \\
&\leq \frac{d}{\eta}\phi_q\bigg(\int_{t_1}^{t_m} w(r)\nabla r\bigg) = \frac{d}{\eta}\eta = d.
\end{aligned}$$

Hence, $F : \overline{P_1(\gamma_1, d)} \to \overline{P_1(\gamma_1, d)}$.

To check condition (ii) of Theorem 2.4, we choose $x_0(t) = Lbt, \qquad t \in [t_1, t_m]$. It is easy to see that $x_0 \in P_1(\gamma_1, \theta_1, \alpha_1, b, Lb, d)$ and $\alpha_1(x_0) > b$, and so

$$\{x \in P_1(\gamma_1, \theta_1, \alpha_1, b, Lb, d) : \alpha_1(x) > b\} \neq \emptyset.$$

Hence, for $t \in [\frac{1}{n}, \frac{n-1}{n}]_{\mathbb{T}}$, $x(t) \in P_1(\gamma_1, \theta_1, \alpha_1, b, Lb, d)$, there is

$$b \leq x(t) \leq Lb, \qquad \mid x^{\Delta}(t) \mid \leq d.$$

Thus, for $t \in [\frac{1}{n}, \frac{n-1}{n}]_{\mathbb{T}}$, by condition (H6) of this theorem, we have

$$f(t, u, v) > \phi_p(nb/\lambda),$$

and combining the condition of α_1 and P_1, we have by (3.8)

$$\begin{aligned}
\alpha_1(Fx) &= \min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}} \mid (Fx)(t) \mid \geq \frac{1}{n}\|Fx\|_{0,\mathbb{T}} = \frac{1}{n}(Fx)(t_m) \\
&= \frac{1}{n}\Bigg[B_0\Bigg(\sum_{i=2}^{m-1} a_i\phi_q\bigg(\int_{t_i}^{t_m} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\bigg)\Bigg) \\
&\quad + \int_{t_1}^{t_m}\phi_q\bigg(\int_{s}^{t_m} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\bigg)\Delta s\Bigg] \\
&> \frac{1}{n}\Bigg[B\Bigg(\sum_{i=2}^{m-1} a_i\phi_q\bigg(\int_{t_i}^{t_i^{\star}} w(r)\nabla r\bigg)\Bigg)\Bigg]\frac{nb}{\lambda} \\
&\geq b,
\end{aligned}$$

i.e.

$$\alpha_1(Fx) > b \quad \text{for all} \quad x \in P_1(\gamma_1, \theta_1, \alpha_1, b, Lb, d).$$

Therefore, condition (i) of Theorem 2.4 is satisfied.

Secondly, by (3.1), we have $x \in P_1(\gamma_1, \alpha_1, b, d)$ with $\theta_1(Fx) > nb$

$$\alpha_1(Fx) \geq \frac{1}{n}\theta_1(Fx) > \frac{1}{n}nb = b.$$

Hence, condition (ii) of Theorem 2.4 is satisfied.

Now we prove that condition (iii) of Theorem 2.4 also holds. Obviously, as $\psi_1(0) = 0 < a$, there holds $0 \notin R(\gamma_1, \psi_1, a, d)$. Assume that $x \in R(\gamma_1, \psi_1, a, d)$ with $\psi_1(x) = a$. Then, from (H7) of this theorem, we find

$$\begin{aligned}
\psi_1(Fx) &= \max_{t\in[t_1,t_m]} \mid (Fx)(t) \mid = \|Fx\|_{0,\mathbb{T}} = (Fx)(t_m) \\
&= B_0\left(\sum_{i=2}^{m-1} a_i\phi_q\left(\int_{t_i}^{t_m} w(r)f(r,x(r),x^{\Delta}(r))\nabla r\right)\right) \\
&\quad + \int_{t_1}^{t_m} \phi_q\left(\int_{s}^{t_m} w(r)f(r,x(r),x^{\Delta}(r))\nabla r\right)\Delta s \\
&< \frac{a}{S}\left[A\left(\sum_{i=2}^{m-1} a_i\phi_q\left(\int_{t_i}^{t_m} w(r)\nabla r\right)\right)\right. \\
&\quad \left. + \int_{t_1}^{t_m} \phi_q\left(\int_{s}^{t_m} w(r)\nabla r\right)\Delta s\right] \\
&\leq a.
\end{aligned}$$

Hence, from here, we get

$$\psi_1(Fx) = \max_{t\in[t_1,t_m]} \mid (Fx)(t) \mid < a,$$

which shows that condition (iii) of Theorem 2.4 is satisfied. On the other hand, for $x \in P_1$, (3.1) holds.

Thus, all the conditions in Theorem 2.4 are met, and so the BVPs (1.1) and (1.2) have at least three positive solutions x_1, x_2, and x_3 such that

$$\begin{aligned}
&\max_{t\in[t_1,t_m]_{\mathbb{T}^k}} \mid x_i^{\Delta}(t) \mid \leq d, && i = 1,2,3; \\
&b < \min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}} \mid x_1(t) \mid, && \max_{t\in[t_1,t_m]} \mid x_1(t) \mid \leq Md, \\
&a < \max_{t\in[t_1,t_m]} \mid x_2(t) \mid, \quad \text{with} && \min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}} \mid x_2(t) \mid < b, \\
&\max_{t\in[t_1,t_m]} \mid x_3(t) \mid < a.
\end{aligned}$$

The proof is complete. □

4. Existence of multiple positive solutions to (1.1) and (1.3)

The method is similar to what we have done in Section 3; therefore, we omit the proof of the main result of this section.

We consider the Banach space E defined as in Section 3 and define a cone $P_2 \subset E$ by

$$P_2 = \left\{x \in E : x(t) \geq 0, \quad x(t_m) + B_1\left(\sum_{i=2}^{m-1} b_i x^{\Delta}(t_i)\right) = 0, x \text{ is concave on } [t_1,t_m]\right\} \subset E.$$

Let the nonnegative continuous concave functional α_2, the nonnegative continuous convex functional θ_2, γ_2, and the nonnegative continuous functional ψ_2 be defined on the cone P_2 by

$$\begin{aligned}\alpha_2(x) &= \min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}} |x(t)|, \qquad \gamma_2(x) = \max_{t\in[t_1,t_m]_{\mathbb{T}^k}} \mid x^{\Delta}(t) \mid, \\ \psi_2(x) &= \theta_2(x) = \max_{t\in[t_1,t_m]} \mid x(t) \mid, \qquad x \in P_2,\end{aligned}$$

where n is defined in Section 3.

Lemma 4.1 *Let* $0 = t_1 < \dfrac{1}{n} < t_2 < \ldots < t_{m-1} = t_m - \dfrac{2}{n} < t_m = 1$. *If (H1) holds, then for* $x \in P_2$, *there exists a constant* $M > 0$ *such that*

$$\max_{t\in[t_1,t_m]} \mid x(t) \mid \leq M \max_{t\in[t_1,t_m]_{\mathbb{T}^k}} \mid x^{\Delta}(t) \mid .$$

By Lemma 4.1 and the concavity of x, for all $x \in P_2$, the functionals defined above hold for the relations

$$\frac{1}{n}\theta_2(x) \leq \alpha_2(x) \leq \theta_2(x), \qquad \|x\|_{1,\mathbb{T}} = \max\{\theta_2(x), \gamma_2(x)\} \leq M\gamma_2(x).$$

Therefore, condition (2.1) of Theorem 2.4 is satisfied.

Lemma 4.2 *If* $y \in C_{ld}[t_1, t_m]$, *then the problem*

$$(\phi_p(x^{\Delta}(t)))^{\nabla} + y(t) = 0, \qquad t_1 < t < t_m, \tag{4.1}$$

$$x^{\Delta}(t_1) = 0, \qquad x(t_m) + B_1\left(\sum_{i=2}^{m-1} b_i x^{\Delta}(t_i)\right) = 0, \tag{4.2}$$

has a unique solution

$$x(t) = B_1\left(\sum_{i=2}^{m-1} b_i\phi_q\left(\int_{t_1}^{t_i} y(r)\nabla r\right)\right) + \int_t^{t_m} \phi_q\left(\int_{t_1}^{s} y(r)\nabla r\right)\Delta s. \tag{4.3}$$

Lemma 4.3 *If* $y \in C_{ld}[t_1, t_m]$ *and* $y \geq 0$, *then the solution of BVPs (4.1) and (4.2) satisfies* $x(t) \geq 0$.

From Lemma 4.2, it is easy to see that BVPs (1.1) and (1.3) have a solution $x = x(t)$ if and only if x solves the equation

$$\begin{aligned}x(t) &= B_1\left(\sum_{i=2}^{m-1} b_i\phi_q\left(\int_{t_1}^{t_i} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\right)\right) \\ &\quad + \int_t^{t_m} \phi_q\left(\int_{t_1}^{s} w(r)f(r, x(r), x^{\Delta}(r))\nabla r\right)\Delta s.\end{aligned}$$

We define the operator $G : P_2 \to P_2$ by

$$
\begin{aligned}
(Gx)(t) \quad &:= \quad B_1\Bigg(\sum_{i=2}^{m-1} b_i\phi_q\bigg(\int_{t_1}^{t_i} w(r)f(r,x(r),x^\Delta(r))\nabla r\bigg)\Bigg) \\
&\quad + \int_t^{t_m} \phi_q\bigg(\int_{t_1}^{s} w(r)f(r,x(r),x^\Delta(r))\nabla r\bigg)\Delta s.
\end{aligned}
$$

Taking the delta derivative of $(Gx)(t)$, we have

$$
(Gx)^\Delta(t) := -\phi_q\bigg(\int_{t_1}^{t} w(r)f(r,x(r),x^\Delta(r))\nabla r\bigg), \quad \text{for} \quad x \in P_2, \quad t \in [t_1, t_m]_{\mathbb{T}^k}.
$$

Lemma 4.4 *Let (H1)–(H4) hold. If conditions 1–3 in the statement of Lemma 3.4 are satisfied, then $FP_2 \subset P_2$ and $F : P_2 \to P_2$ is completely continuous.*

Moreover, we can prove the following result:

$$
\min_{t\in[\frac{1}{n},\frac{n-1}{n}]_\mathbb{T}} (Gx)(t) \geq \frac{1}{n}\|Gx\|_{0,\mathbb{T}} = \frac{1}{n}(Gx)(t_1).
$$

Let

$$
\eta = \phi_q\left(\int_{t_1}^{t_m} w(r)\nabla r\right), \qquad \bar{\lambda}_i = B\Bigg(\sum_{i=2}^{m-1} b_i\phi_q\bigg(\int_{t_i}^{t_i^\star} w(r)\nabla r\bigg)\Bigg),
$$

$$
\bar{\lambda} = \min_{i\in[t_1,t_{m-1}]}\{\bar{\lambda}_i\}, \qquad t_i^\star = \frac{t_i + t_{i+1}}{2} \quad (i = 1, 2, \ldots, m-1),
$$

$$
\bar{S}_i = A\Bigg(\sum_{i=2}^{m-1} b_i\phi_q\bigg(\int_{t_1}^{t_i} w(r)\nabla r\bigg)\Bigg) + \int_{t_1}^{t_m} \phi_q\bigg(\int_{t_1}^{s} w(r)\nabla r\bigg)\Delta s, \quad \bar{S} = \max_{i\in[t_1,t_{m-1}]}\{\bar{S}_i\},
$$

$$
L = \frac{n}{2}\left[1 + \left(1 - \sum_{i=2}^{m-1} a_i\right)\right], \quad 0 = t_1 < \frac{1}{n} < t_2 < \ldots < t_{m-1} = t_m - \frac{2}{n} < t_m = 1.
$$

We have the following result.

Theorem 4.5 *Suppose that (H1)–(H4) hold. Let $0 < a < b \leq \frac{2Md}{L}$, and assume that f satisfies the following conditions:*

(H8) $f(t,u,v) \leq \phi_p(d/\eta)$, *for* $(t,u,v) \in [t_1,t_m] \times [0, Md] \times [-d,d]$;

(H9) $f(t,u,v) > \phi_p(nb/\bar{\lambda})$, *for* $(t,u,v) \in [\frac{1}{n},\frac{n-1}{n}]_\mathbb{T} \times [b, Lb] \times [-d,d]$;

(H10) $f(t,u,v) < \phi_p(a/\bar{S})$, *for* $(t,u,v) \in [t_1,t_m] \times [0,a] \times [-d,d]$;

then the BVPs (1.1) and (1.3) have at least three positive solutions x_1, x_2, and x_3 such that

$$\max_{t\in[t_1,t_m]_{\mathbb{T}^k}} \mid x_i^{\Delta}(t) \mid\leq d, \qquad i=1,2,3;$$

$$b<\min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}} \mid x_1(t)\mid, \qquad \max_{t\in[t_1,t_m]} \mid x_1(t)\mid\leq Md,$$

$$a<\max_{t\in[t_1,t_m]} \mid x_2(t)\mid, \quad \textit{with} \quad \min_{t\in[\frac{1}{n},\frac{n-1}{n}]_{\mathbb{T}}} \mid x_2(t)\mid< b,$$

$$\max_{t\in[t_1,t_m]} \mid x_3(t)\mid< a.$$

In the following section, we give two examples to explain our results.

5. Examples

Example 5.1. Let $\mathbb{T}=\{0,\frac{4}{5}\}\cup\{\frac{1}{5^n}:n\in\mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers numbers set. If we choose $t_1=0, t_2=\frac{1}{2},\ t_m=1, p=3, a_2=\frac{1}{2}, A=B=\frac{1}{2}$, then $\phi_p(x)=x^2$. Suppose $w(t)\equiv 1$ and consider the following BVP on time scales

$$\Big(\phi_p(x^{\Delta}(t))\Big)^{\nabla}+f(t,x(t),x^{\Delta}(t))=0, \qquad t_1<t<t_m, \tag{5.1}$$

$$x(0)-\frac{1}{2}\Big(\frac{1}{2}x^{\Delta}\Big(\frac{1}{2}\Big)\Big)=0, \qquad x^{\Delta}(1)=0, \tag{5.2}$$

where

$$f(t,u,v)=\begin{cases} \frac{1}{5}t+\frac{60^{14}}{73^6}u^{13}+\frac{1}{1000}\Big(\frac{73^6}{60^{15}}v\Big)^2, & \text{if } \ u\leq\frac{1}{16}\times\frac{74^{\frac{6}{11}}}{60^{\frac{1}{11}}},\\ \frac{1}{5}t+\frac{60^{14}}{73^6}\times\Big[\frac{1}{16}\times\frac{74^{\frac{6}{11}}}{60^{\frac{1}{11}}}\Big]^{13}+\frac{1}{1000}\Big(\frac{73^6}{60^{15}}v\Big)^2, & \text{if } \ u>\frac{1}{16}\times\frac{74^{\frac{6}{11}}}{60^{\frac{1}{11}}}. \end{cases}$$

By simple calculations, we have

$$n=5>\max\Big\{\frac{1}{t_2},\frac{2}{1-t_2}\Big\}, \qquad M=\frac{1}{2}a_2+t_m-t_1=\frac{5}{4}, \qquad L=\frac{n}{2}[1+(1-a_2)]=\frac{15}{4},$$

$$\eta=\phi_q\left(\int_0^1\nabla r\right)=1, \quad \lambda_1=\frac{1}{2}\left(a_2\phi_q\left(\int_0^{1/4}\nabla r\right)\right), \quad \lambda_2=\frac{1}{2}\left(a_2\phi_q\left(\int_{1/2}^{3/4}\nabla r\right)\right),$$

$$\lambda=\min\{\lambda_1,\lambda_2\}=\frac{1}{8}, \quad S_1=\frac{1}{2}\left(a_2\phi_q\left(\int_0^1\nabla r\right)\right)+\int_0^1\phi_q\left(\int_s^1\nabla r\right)\Delta s,$$

$$S_2=\frac{1}{2}\left(a_2\phi_q\left(\int_{1/2}^1\nabla r\right)\right)+\int_0^1\phi_q\left(\int_s^1\nabla r\right)\Delta s,\qquad S=\max\{S_1,S_2\}=\frac{11}{12}.$$

If we take

$$a=\frac{72^{\frac{6}{11}}}{4^{\frac{1}{4}}\times 60^{\frac{14}{11}}},\quad b=\frac{74^{\frac{6}{11}}}{60^{\frac{12}{11}}},\quad d=\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}},$$

then it is easy to see that $0<a<b<\frac{2Md}{L}$, and f satisfies

$$\begin{aligned}
f(t,u,v)&\leq\frac{1}{5}+\frac{60^{14}}{73^6}\times\frac{1}{16}\times\frac{74^{\frac{6}{11}}}{60^{\frac{1}{11}}}+\frac{1}{1000}\approx 2.33341\times 10^{13}\\
&<\phi_3\left(\frac{d}{\eta}\right)=\left[\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}}\right]^2\approx 6.14129\times 10^{21},\\
&\text{for } (t,u,v)\in[0,1]\times\left[0,\frac{5}{4}\times\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}}\right]\\
&\times\left[-\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}},\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}}\right];\\
f(t,u,v)&>\frac{1}{5}+\frac{60^{14}}{73^6}\times\frac{74^{\frac{78}{11}}}{60^{\frac{156}{11}}}+\frac{1}{1000}\approx 56.6056\\
&>\phi_3\left(\frac{nb}{\lambda}\right)=\left[\frac{5\times\frac{74^{\frac{6}{11}}}{60^{\frac{12}{11}}}}{\frac{1}{8}}\right]^2\approx 23.1036,\\
&\text{for } (t,u,v)\in\left[\frac{1}{5},\frac{4}{5}\right]\times\left[\frac{74^{\frac{6}{11}}}{60^{\frac{12}{11}}},\frac{15}{4}\times\frac{74^{\frac{6}{11}}}{60^{\frac{12}{11}}}\right]\\
&\times\left[-\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}},\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}}\right];
\end{aligned}$$

$$\begin{aligned}
f(t,u,v)&<\frac{60^{14}}{73^6}\times\frac{72^{\frac{78}{11}}}{4^{\frac{13}{4}}\times 60^{\frac{182}{11}}}+\frac{1}{1000}\approx 0.00103216\\
&<\phi_3\left(\frac{a}{S}\right)=\left[\frac{\frac{72^{\frac{6}{11}}}{4^{\frac{1}{4}}\times 60^{\frac{14}{11}}}}{\frac{11}{12}}\right]^2\approx 0.00188159,\\
&\text{for } (t,u,v)\in[0,1]\times\left[0,\frac{72^{\frac{6}{11}}}{4^{\frac{1}{4}}\times 60^{\frac{14}{11}}}\right]\\
&\times\left[-\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}},\left(\frac{15}{4}\right)^{13}\times\frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}}\right].
\end{aligned}$$

Therefore, all the conditions of Theorem 3.5 are satisfied. Thus, by Theorem 3.5, the problems (5.1) and (5.2) have at least three positive solutions $x_1, x_2,$ and x_3 such that

$$\max_{t\in[0,1]_{\mathbb{T}^k}} | x_i^{\Delta}(t) |\leq \left(\frac{15}{4}\right)^{13} \times \frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}}, \qquad i = 1, 2, 3;$$

$$\frac{74^{\frac{6}{11}}}{60^{\frac{12}{11}}} < \min_{t\in[\frac{1}{5},\frac{4}{5}]_{\mathbb{T}}} | x_1(t) |, \qquad \max_{t\in[0,1]} | x_1(t) |\leq \frac{5}{4} \times \left(\frac{15}{4}\right)^{13} \times \frac{74^{\frac{24}{11}}}{60^{\frac{4}{11}}},$$

$$\frac{72^{\frac{6}{11}}}{4^{\frac{1}{4}} \times 60^{\frac{14}{11}}} < \max_{t\in[0,1]} | x_2(t) |, \quad \text{with} \quad \min_{t\in[\frac{1}{5},\frac{4}{5}]_{\mathbb{T}}} | x_2(t) |< \frac{74^{\frac{6}{11}}}{60^{\frac{12}{11}}},$$

$$\max_{t\in[0,1]} | x_3(t) |< \frac{72^{\frac{6}{11}}}{4^{\frac{1}{4}} \times 60^{\frac{14}{11}}}.$$

Example 5.2. Let $\mathbb{T} = \{1-(\frac{1}{2})^{\mathbb{N}_0}\}\cup\{\frac{1}{3}, 1\}$; $\mathbb{N}_0$ denotes the set of all nonnegative integers. Take $a_2 = \frac{1}{2},\ t_1 = 0,\ t_2 = \frac{1}{2},\ \ t_m = 1,\ A = B = \frac{1}{2},\ p = q = 2,$ and $w(t) \equiv 1$. Consider the following BVP:

$$\Big(\phi_p(x^{\Delta}(t))\Big)^{\nabla} + f(t, x(t), x^{\Delta}(t)) = 0, \qquad t_1 < t < t_m, \tag{5.3}$$

$$x(0) - \frac{1}{2}\Big(\frac{1}{2}x^{\Delta}\Big(\frac{1}{2}\Big)\Big) = 0, \qquad x^{\Delta}(1) = 0, \tag{5.4}$$

where

$$f(t,u,v) = \begin{cases} \frac{t}{1000} + 4000u^3 + \left(\frac{v}{100000}\right)^3, & u \leq \frac{1}{3}, \\ \frac{t}{1000} + 148 + \left(\frac{v}{100000}\right)^3, & u > \frac{1}{3}. \end{cases}$$

By simple calculations, we have

$$n = 5 > \max\left\{\frac{1}{t_2}, \frac{2}{1-t_2}\right\}, \qquad M = \frac{1}{2}a_2 + t_m - t_1 = \frac{5}{4}, \qquad L = \frac{n}{2}[1 + (1 - a_2)] = \frac{15}{4},$$

$$\eta = \phi_q\left(\int_0^1 \nabla r\right) = 1, \qquad \lambda_1 = \frac{1}{2}\left(a_2\phi_q\left(\int_0^{1/4} \nabla r\right)\right), \qquad \lambda_2 = \frac{1}{2}\left(a_2\phi_q\left(\int_{1/2}^{3/4} \nabla r\right)\right),$$

$$\lambda = \min\{\lambda_1, \lambda_2\} = \frac{1}{16}, \qquad S_1 = \frac{1}{2}\left(a_2\phi_q\left(\int_0^1 \nabla r\right)\right) + \int_0^1 \phi_q\left(\int_s^1 \nabla r\right)\Delta s,$$

$$S_2=\frac{1}{2}\left(a_2\phi_q\left(\int_{1/2}^{1}\nabla r\right)\right)+\int_0^1\phi_q\left(\int_s^1\nabla r\right)\Delta s,\qquad S=\max\{S_1,S_2\}=\frac{3}{4}.$$

If we take $a=\frac{1}{64}$, $b=\frac{1}{6}$, $d=200$, then it is easy to see that $0<a<b<\frac{2Md}{L}$, and f satisfies

$$f(t,u,v)\le\phi_p\Big(\frac{d}{\eta}\Big)=200,\quad \text{for } 0\le t\le 1,\quad 0\le u\le 250,\quad |\, v\,|\le 200;$$

$$f(t,u,v)>\phi_p\Big(\frac{nb}{\lambda}\Big)=\frac{40}{3},\quad \text{for}\ \ \frac{1}{5}\le t\le\frac{4}{5},\quad \frac{1}{6}\le u\le\frac{5}{8},\quad |\, v\,|\le 200;$$

$$f(t,u,v)<\phi_p\Big(\frac{a}{S}\Big)\approx 0.0208333,\quad \text{for}\ \ 0\le t\le 1,\quad 0\le u\le\frac{1}{64},\quad |\, v\,|\le 200.$$

Hence, by Theorem 3.5, we have that the BVPs (5.3) and (5.4) have at least three positive solutions $x_1,\ x_2,\ x_3$ such that

$$\max_{t\in[0,1]_{\mathbb{T}^k}}|\, x_i^{\Delta}(t)\,|\le 200,\qquad i=1,2,3;$$

$$\frac{1}{6}<\min_{t\in[\frac{1}{5},\frac{4}{5}]_{\mathbb{T}}}|\, x_1(t)\,|,\qquad \max_{t\in[0,1]}|\, x_1(t)\,|\le 250,$$

$$\frac{1}{64}<\max_{t\in[0,1]}|\, x_2(t)\,|,\quad\text{with}\quad \min_{t\in[\frac{1}{5},\frac{4}{5}]_{\mathbb{T}}}|\, x_2(t)\,|<\frac{1}{6},$$

$$\max_{t\in[0,1]}|\, x_3(t)\,|<\frac{1}{64}.$$

6. Conclusions

1. One may establish new criteria from the proofs of our results for p-Laplacian boundary value problems (1.1) (1.2) and (1.1) (1.3). The details are left to the reader.

2. The results of our paper are new for the discrete case ($\mathbb{T}=\mathbb{Z}$) and the continuous case ($\mathbb{T}=\mathbb{R}$). The formulation of our results for both cases is left to the reader.

Finally, we remark (we leave the details to the reader) that similar ideas could be used to discuss the more general problem

$$(\phi_p(x^{\Delta}(t)))^{\nabla}+w(t)f(t,x(t),x^{\Delta}(t))=0,\qquad t_1<t<t_m,$$

$$x(t_1)-B_0\Big(\sum_{i=2}^{m-1}a_ix^{\Delta}(t_i)\Big)=0,\qquad x^{\Delta}(t_m)=\sum_{i=2}^{m-1}b_ix(t_i),$$

or

$$x^{\Delta}(t_1)=\sum_{i=2}^{m-1}a_ix(t_i),\qquad x(t_m)+B_1\Big(\sum_{i=2}^{m-1}b_ix^{\Delta}(t_i)\Big)=0.$$

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