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EXISTENCE OF POSITIVE SOLUTIONS FOR p-LAPLACIAN AN m-POINT BOUNDARY VALUE PROBLEM INVOLVING THE DERIVATIVE ON TIME SCALES

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Abstract. We are interested in the existence of positive solutions for the p-Laplacian dynamic equation on time scales,

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t, u(t), u^{\Delta}(t)) = 0, \quad t \in (0, T)_{\mathbb{T}},$$

subject to the multipoint boundary condition,

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u^{\Delta}(T) = 0,$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in [0, T]_{\mathbb{T}}$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$. By using fixed point theorems, we prove the existence of at least three non-negatvie solutions, two of them positive, to the above boundary value problem. The interesting point is the nonlinear term f is involved with the first order derivative explicitly. An example is given to illustrate the main result.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced and developed by Aulbach and Hilger [14] in 1988. It has been created in order to unify continuous and discrete analysis, and it allows a simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. Further, the study of time scales has led to several important applications, e.g., in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing, and epidemic models, see [15, 20, 24].

Recently, much attention has been paid to the existence of multipoint positive solutions of boundary value problems (BVPs) on time scales. When the nonlinear term f does not depend on the first order derivative, many researchers have studied multipoint boundary conditions on time scales; see [1, 6, 8, 9, 10, 13, 16, 18, 19, 22, 25]. However, little work has done on the existence of positive solutions for multipoint BVP on time scales when the nonlinear term is involved in first order derivative explicitly; see [11, 21, 23].

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There is recent work in fixed point applications using convex and concave functionals in which there is nonlinear dependence on higher order derivatives; see [2, 17].

Motivated by all the above works, we are interested in the existence of at least three non-negative solutions, two of them positive, for p-Laplacian dynamic equation on time scales,

$$(\phi_p(u^\Delta(t)))^\nabla + a(t)f(t, u(t), u^\Delta(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}, \tag{1.1}$$

subject to boundary condition

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u^\Delta(T) = 0, \tag{1.2}$$

where $\phi_p(u)$ is p-Laplacian operator; i.e., $\phi_p(s) = |s|^{p-2}s$, for $p > 1$, with $(\phi_p)^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. The usual notation and terminology for time scales as can be found in [4, 5], will be used here. The interesting point is that the nonlinear term f is involved with the first order derivative explicitly. Our main results will depend on an application of a generalization of the Leggett-Williams fixed point theorem due to Bai and Ge. An example is also given to illustrate the main results. The results are even new for the special cases of difference equations and differential equations, as well as in the general time scale setting. We shall prove that the BVP (1.1) and (1.2) has at least three non-negative solutions.

Throughout the paper, we will suppose that the following conditions are satisfied:

- (H1) $0, T \in \mathbb{T}$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$, $\xi_i \in \mathbb{T}$, $\alpha_i \in [0, \infty)$ for $i = 1, \dots, m-2$, and $1 - \sum_{i=1}^{m-2} \alpha_i > 0$;
- (H2) $\eta = \min\{t \in \mathbb{T} : \frac{T}{2} \le t < T\}$ exists;
- (H3) $a(t) \in C_{ld}([0, T]_{\mathbb{T}}, [0, \infty))$ with $0 < \int_\eta^T a(t)\nabla t < \infty$;
- (H4) $f : (0, T)_{\mathbb{T}} \times [0, \infty) \times \mathbb{R} \to [0, \infty)$ is continuous;
- (H5) $a(t)f(t, 0, 0) \not\equiv 0$, $f(t, 0, 0) \ge 0$ on $[0, T]_{\mathbb{T}}$.

The rest of this article is arranged as follows. We state some definitions, notation, lemmas and prove several preliminary results in Section 2. The main theorem on the existence of at least three non-negative solutions and its proof are presented in Section 3. In section 4, we give an example to demonstrate our results.

2. Preliminaries

In this section, we provide some background materials from theory of cones in Banach spaces. The following definitions can be found in the book by Deimling [7] as well as in the book by Guo and Lakshmikantham [12].

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following two conditions:

- (i) $x \in P$, $\lambda \ge 0$ imply $\lambda x \in P$;
- (ii) $x \in P$, $-x \in P$ imply $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \le y$ if and only if $y - x \in P$.

Definition 2.2. A map ψ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\psi : P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \ge t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Similarly, we say the map α is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Let ψ be a nonnegative continuous concave functional on P, and α and β be nonnegative continuous convex functionals on P. For nonnegative real numbers r, a and l, we define the following convex sets.

$$\begin{gathered}
P(\alpha, r; \beta, l) = \{u \in P : \alpha(u) < r, \beta(u) < l\}, \\
\bar{P}(\alpha, r; \beta, l) = \{u \in P : \alpha(u) \le r, \beta(u) \le l\}, \\
P(\alpha, r; \beta, l; \psi, a) = \{u \in P : \alpha(u) < r, \beta(u) < l, \psi(u) > a\}, \\
\bar{P}(\alpha, r; \beta, l; \psi, a) = \{u \in P : \alpha(u) \le r, \beta(u) \le l, \psi(u) \ge a\}.
\end{gathered}$$

To prove our main results, we need the following fixed point theorem, which comes from Bai and Ge in [3].

Lemma 2.3 ([3]). *Let P be a cone in a real Banach space E. Assume that constants r_1, b, d, r_2, l_1 and l_2 satisfy $0 < r_1 < b < d \le r_2$ and $0 < l_1 \le l_2$. If there exist two nonnegative continuous convex functionals α and β on P and a nonnegative continuous concave functional ψ on P such that*

- (A1) *there exists $M > 0$ such that $\|u\| \le M \max\{\alpha(u), \beta(u)\}$ for all $u \in P$;*
- (A2) *$P(\alpha, r; \beta, l) \ne \emptyset$ for any $r > 0$ and $l > 0$;*
- (A3) *$\psi(u) \le \alpha(u)$ for all $u \in \bar{P}(\alpha, r_2; \beta, l_2)$;*

and if $F : \bar{P}(\alpha, r_2; \beta, l_2) \to (\alpha, r_2; \beta, l_2)$ is completely continuous operator, which satisfies

- (B1) *$\{u \in \bar{P}(\alpha, d; \beta, l_2; \psi, b) : \psi(u) > b\} \ne \emptyset$, $\psi(Fu) > b$ for $u \in \bar{P}(\alpha, d; \beta, l_2; \psi, b)$;*
- (B2) *$\alpha(Fu) < r_1$, $\beta(Fu) < l_1$ for $u \in \bar{P}(\alpha, r_1; \beta, l_1)$;*
- (B3) *$\psi(Fu) > b$ for $u \in \bar{P}(\alpha, r_2; \beta, l_2; \psi, b)$ with $\alpha(Fu) > d$.*

Then F has at least three different fixed points u_1, u_2 and u_3 in $\bar{P}(\alpha, r_2; \beta, l_2)$ with

$$\begin{gathered}
u_1 \in P(\alpha, r_1; \beta, l_1), \quad u_2 \in \{\bar{P}(\alpha, r_2; \beta, l_2; \psi, b) : \psi(u) > b\}, \\
u_3 \in \bar{P}(\alpha, r_2; \beta, l_2) \setminus \big(\bar{P}(\alpha, r_2; \beta, l_2; \psi, b) \cup \bar{P}(\alpha, r_1; \beta, l_1)\big).
\end{gathered}$$

Let the Banach space

$$E = \{u : [0,T]_{\mathbb{T}} \to \mathbb{R} : u \text{ is } \Delta\text{-differentiable and } u^{\Delta} \text{ is ld-continuous on } [0,T]_{\mathbb{T}}\}$$

be endowed with norm

$$\|u\| = \max\Big\{ \sup_{t \in [0,T]_{\mathbb{T}}} |u(t)|, \sup_{t \in [0,T]_{\mathbb{T}}} |u^{\Delta}(t)| \Big\}.$$

Define

$$P = \{u \in E : u(t) \ge 0, u^{\Delta}(t) \ge 0, \text{ and } u(t) \text{ is concave on } [0,T]_{\mathbb{T}}\}.$$

Clearly, P is a cone.

Lemma 2.4. *If $\sum_{i=1}^{m-2}\alpha_i \neq 1$, then for $h \in C_{ld}[0,T]_{\mathbb{T}}$ and $h \geq 0$,*

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + h(t) = 0, \quad t \in (0,T)_{\mathbb{T}}, \tag{2.1}$$

$$u(0) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i), \quad u^{\Delta}(T) = 0 \tag{2.2}$$

has the unique solution

$$u(t) = \int_0^t \phi_q\Big(\int_s^T h(\tau)\nabla\tau\Big)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T h(\tau)\nabla\tau\Big)\Delta s. \tag{2.3}$$

Moreover, if $h(t) \geq 0$ on $[0,T]_{\mathbb{T}}$ and (H1) *is satisfied, then $u(t) \geq 0$ on $[0,T]_{\mathbb{T}}$.*

Proof. Let u be as in (2.3), taking the delta derivative of (2.3), we have

$$u^{\Delta}(t) = \phi_q\Big(\int_t^T h(\tau)\nabla\tau\Big),$$

moreover, we obtain

$$\phi_p(u^{\Delta}(t)) = \int_t^T h(\tau)\nabla\tau,$$

taking the nabla derivative of this expression yields $(\phi_p(u^{\Delta}(t)))^{\nabla} = -h(t)$. Routine calculations verify that u satisfies the boundary value conditions in (2.2), so that u given in (2.3) is a solution of (2.1) and (2.2). It is easy to see that BVP $(\phi_p(u^{\Delta}))^{\nabla} = 0$, $u(0) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i)$, $u^{\Delta}(T) = 0$ has only the trivial solution. Thus u in (2.3) is the unique solution of (2.1) and (2.2). The proof is complete. $\square$

Lemma 2.5. *The solution of BVP* (2.1) *and* (2.2) *satisfies $u(t) \geq 0$, for $t \in [0,T]_{\mathbb{T}}$.*

Proof. Let

$$\varphi(s) = \phi_q\Big(\int_s^T h(\tau)\nabla\tau\Big).$$

Since $\int_s^T h(\tau)\nabla\tau \geq 0$, it follows that $\varphi(s) \geq 0$. According to Lemma 2.4, we obtain

$$u(0) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi(s)\Delta s \geq 0,$$

$$u(T) = \int_0^T \varphi(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi(s)\Delta s \geq 0.$$

If $t \in (0,T)$, we have

$$u(t) = \int_0^t \varphi(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi(s)\Delta s \geq 0.$$

So $u(t) \geq 0$ for $t \in [0,T]$. $\square$

Lemma 2.6. *The solution of* (1.1) *and* (1.2) *satisfies*

$$\inf_{t\in[0,T]_{\mathbb{T}}} u(t) \geq \gamma\|u\|$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{T-\sum_{i=1}^{m-2}\alpha_i(T-\xi_i)}.$$

Proof. Clearly $u^{\Delta}(t) = \varphi(t) \geq 0$. This implies that

$$\min_{t\in[0,T]} u(t) = u(0), \quad \|u\| = u(T).$$

It is easy to see that $u^{\Delta}(t_2) \leq u^{\Delta}(t_1)$ for any $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$. Hence $u^{\Delta}(t)$ is a decreasing function on $[0,T]$. This means that graph of $u(t)$ is concave down on $(0,T)$. For each $i \in \{1,2,\dots,m-2\}$, we have

$$\frac{u(T)-u(0)}{T-0} \geq \frac{u(T)-u(\xi_i)}{T-\xi_i},$$

i.e.,

$$Tu(\xi_i) - \xi_i u(T) \geq (T-\xi_i)u(0),$$

so that

$$T\sum_{i=1}^{m-2}\alpha_i u(\xi_i) - \sum_{i=1}^{m-2}\alpha_i\xi_i u(T) \geq \sum_{i=1}^{m-2}\alpha_i(T-\xi_i)u(0).$$

With the boundary condition $u(0) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i)$, we have

$$u(0) \geq \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{T-\sum_{i=1}^{m-2}\alpha_i(T-\xi_i)}u(T).$$

This completes the proof. □

Define the operator $F : P \to E$ by

$$\begin{aligned}(Fu)(t) &= \int_0^t \phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s\end{aligned}$$

for $t \in [0,T]_{\mathbb{T}}$. By the definition of F, the monotonicity of $\phi_q(u)$ and assumption of (H1)-(H5), it is easy to see that for each $u \in P$, $Fu \in P$ and $Fu(T)$ is the maximum value of $Fu(t)$. Moreover, by direct calculation, we obtain that each fixed point of the operator F in P is a positive solution of (1.1) and (1.2). It is easy to see that $F : P \to P$ is completely continuous.

3. Existence of positive solutions

For $u \in P$ we define

$$\alpha(u) = \max_{t\in[0,T]_{\mathbb{T}}}|u(t)| = u(T), \quad \beta(u) = \sup_{t\in[0,T]_{\mathbb{T}}}|u^{\Delta}(t)| = u^{\Delta}(0),$$
$$\psi(u) = \min_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta).$$

It is easy to see that $\alpha, \beta : P \to [0,\infty)$ are nonnegative continuous convex functionals with $\|u\| = \max\{\alpha(u), \beta(u)\}$; $\psi : P \to [0,\infty)$ is nonnegative concave functional. We have $\psi(u) \leq \alpha(u)$ for $u \in P$ and assumptions (A1), (A2) and (A3) in Lemma 2.3 hold.

For notational convenience, we denote

$$M = \int_0^{\eta}\phi_q\Big(\int_{\eta}^T a(\tau)\nabla\tau\Big)\Delta s,$$

$$N=\int_0^T \phi_q\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s,$$
$$L=\phi_q\Big(\int_0^T a(\tau)\nabla\tau\Big).$$

Theorem 3.1. *Assume that* (H1)–(H5) *hold, and there exists* $0<r_1<b<2b\le r_2$, $0<l_1\le l_2$ *such that* $\frac{b}{M}\le\min\{r_2/N,l_2/L\}$. *If f satisfies the following three conditions:*

(i) $f(t,w,v)\le\min\{\phi_p(r_2/N),\phi_p(l_2/L)\}$ *for* $(t,w,v)\in[0,T]_{\mathbb{T}}\times[0,r_2]\times[-l_2,l_2]$;
(ii) $f(t,w,v)>\phi_p(b/M)$ *for* $(t,w,v)\in[\eta,T]_{\mathbb{T}}\times[b,2b]\times[-l_2,l_2]$;
(iii) $f(t,w,v)<\min\{\phi_p(r_1/N),\phi_p(l_1/L)\}$ *for* $(t,w,v)\in[0,T]_{\mathbb{T}}\times[0,r_1]\times[-l_1,l_1]$;

then BVP (1.1) *and* (1.2) *has at least three three non-negative solutions, two of them positive, u_1,u_2,u_3, which satisfy*

$$\max_{t\in[0,T]_{\mathbb{T}}}\{u_1(t)\}<r_1,\quad \sup_{t\in[0,T]_{\mathbb{T}}}|u_1^{\Delta}(t)|<l_1;$$
$$b<\min_{t\in[\eta,T]_{\mathbb{T}}}\{u_2(t)\}\le\max_{t\in[0,T]_{\mathbb{T}}}\{u_2(t)\}\le r_2,\quad \sup_{t\in[0,T]_{\mathbb{T}}}|u_2^{\Delta}(t)|\le l_2;$$
$$\min_{t\in[\eta,T]_{\mathbb{T}}}\{u_3(t)\}<b,\quad r_1<\max_{t\in[0,T]_{\mathbb{T}}}\{u_3(t)\}<2b,\quad l_1<\sup_{t\in[0,T]_{\mathbb{T}}}|u_3^{\Delta}(t)|\le l_2.$$

Proof. To show Lemma 2.3 holds, it is sufficient to show that conditions in Lemma 2.3 are satisfied with respect to operator F. We first prove that if the assumption (i) is satisfied, then $F:\bar{P}(\alpha,r_2;\beta,l_2)\to\bar{P}(\alpha,r_2;\beta,l_2)$. If $u\in\bar{P}(\alpha,r_2;\beta,l_2)$, then

$$\alpha(u)=\max_{t\in[0,T]_{\mathbb{T}}}|u(t)|\le r_2,\quad \beta(u)=\sup_{t\in[0,T]_{\mathbb{T}}}|u^{\Delta}(t)|\le l_2$$

and assumption (i) implies

$$f(t,u(t),u^{\Delta}(t))\le\min\Big\{\phi_p(r_2/N),\phi_p(l_2/L)\Big\},t\in[0,T]_{\mathbb{T}}.$$

For $u\in P$, there is $Fu\in P$, therefore

$$\begin{aligned}
\alpha(Fu)&=\max_{t\in[0,T]_{\mathbb{T}}}|(Fu)(t)|\\
&=\max_{t\in[0,T]_{\mathbb{T}}}\Big|\int_0^t\phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s\Big|\\
&=\int_0^T\phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s\\
&<\int_0^T\phi_q\Big(\int_s^T a(\tau)\phi_p(r_2/N)\nabla\tau\Big)\Delta s\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T a(\tau)\phi_p(r_2/N)\nabla\tau\Big)\Delta s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{r_2}{N}\Big[\int_0^T \phi_q\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T a(\tau)\nabla\tau\Big)\Delta s\Big] = r_2
\end{aligned}
$$

and

$$
\begin{aligned}
\beta(Fu) &= \sup_{t\in[0,T]_{\mathbb{T}}} |(Fu)^{\Delta}(t)| \\
&= \sup_{t\in[0,T]_{\mathbb{T}}} \Big|\phi_q\Big(\int_t^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Big| \\
&= \phi_q\Big(\int_0^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big) \\
&\le \phi_q\Big(\int_0^T a(\tau)\phi_p(l_2/L)\nabla\tau\Big) \\
&= \frac{l_2}{L}\phi_q\Big(\int_0^T a(\tau)\nabla\tau\Big) = l_2.
\end{aligned}
$$

Therefore $F:\bar{P}(\alpha,r_2;\beta,l_2)\to\bar{P}(\alpha,r_2;\beta,l_2)$. Similarly, if $u\in\bar{P}(\alpha,r_1;\beta,l_1)$, then the assumption (iii) implies

$$
f(t,u(t),u^{\Delta}(t)) < \min\{\phi_p(r_1/N),\phi_p(l_1/L)\} \quad \text{for } t\in[0,T]_{\mathbb{T}}.
$$

We can get that $F:\bar{P}(\alpha,r_1;\beta,l_1)\to P(\alpha,r_1;\beta,l_1)$. So condition (B2) of Lemma 2.3 is satisfied.

To prove that condition (B1) of Lemma 2.3 holds. We choose $u(t)=2b$ for $t\in[0,T]_{\mathbb{T}}$. It is obvious that $u(t)=2b\in\bar{P}(\alpha,2b;\beta,l_2;\psi,b)$ and $\psi(u)=2b>b$, and consequently

$$
\{u\in\bar{P}(\alpha,2b;\beta,l_2;\psi,b):\psi(u)>b\}\neq\emptyset.
$$

So, for $u\in\bar{P}(\alpha,2b;\beta,l_2;\psi,b)$, there are $b\le u(t)\le 2b$ and $|u^{\Delta}(t)|\le l_2$ for $t\in[\eta,T]_{\mathbb{T}}$. Thus from the assumption (ii) we have

$$
f(t,u(t),u^{\Delta}(t)) > \phi_p(b/M) \quad \text{for } t\in[\eta,T]_{\mathbb{T}}.
$$

From the definition of the functional ψ we see that

$$
\begin{aligned}
\psi(Fu) &= \min_{t\in[\eta,T]_{\mathbb{T}}} Fu(t) = Fu(\eta) \\
&= \int_0^{\eta}\phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\Big(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s \\
&\ge \int_0^{\eta}\phi_q\Big(\int_{\eta}^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big)\Delta s \\
&> \int_0^{\eta}\phi_q\Big(\int_{\eta}^T a(\tau)\phi_p(b/M)\nabla\tau\Big)\Delta s \\
&= \frac{b}{M}\int_0^{\eta}\phi_q\Big(\int_{\eta}^T a(\tau)\nabla\tau\Big)\Delta s = b.
\end{aligned}
$$

So, we obtain $\psi(Fu) > b$ for $u \in \bar{P}(\alpha, 2b; \beta, l_2; \psi, b)$, and condition (B1) of Lemma 2.3 holds.

Finally, we prove that condition (B3) of Lemma 2.3 holds. If $u \in \bar{P}(\alpha, r_2; \beta, l_2; \psi, b)$ and $\alpha(Fu) > 2b$, we have

$$\psi(Fu) = \min_{t\in[\eta,T]_{\mathbb{T}}} Fu(t) = Fu(\eta) \geq \frac{\eta}{T} \max_{t\in[0,T]_{\mathbb{T}}} Fu(t) \geq \frac{1}{2}\alpha(Fu) > b.$$

Hence, condition (B3) of Lemma 2.3 is satisfied. Then using Lemma 2.3 and the assumption that $f(t,0,0) \not\equiv 0$ on $[0,T]_{\mathbb{T}}$, we find that there exist at least three non-negative solutions of (1.1) and (1.2) such that

$$\begin{gathered} u_1 \in P(\alpha, r_1; \beta, l_1), \quad u_2 \in \{P(\alpha, r_2; \beta, l_2; \psi, b) | \psi(u) > b\}, \\ u_3 \in \bar{P}(\alpha, r_2; \beta, l_2) \setminus \Big(\bar{P}(\alpha, r_2; \beta, l_2; \psi, b) \cup \bar{P}(\alpha, r_1; \beta, l_1)\Big). \end{gathered}$$

Otherwise, as u_3 satisfies $\alpha(u_3) \leq 2\psi(u_3)$, we have $\max_{t\in[0,T]_{\mathbb{T}}} u_3(t) < 2b$. □

In the following section, we now give an example to illustrate our results.

4. An example

Let $\mathbb{T} = \{1 - (\frac{1}{2})^{\mathbb{N}_0}\} \cup [1,2]$, and let $\mathbb{N}_0$ denote the set of nonnegative integers. Take $\alpha_1 = 1/2$, $\alpha_2 = 1/6$, $\xi_1 = 1/4$, $\xi_2 = 3/4$, $T = 2$, $p = q = 2$, and $a(t) \equiv 1$ for $t \in [0,T]_{\mathbb{T}}$. Consider the BVP

$$\Big(u^{\Delta}(t)\Big)^{\nabla} + f(t, u(t), u^{\Delta}(t)) = 0, \quad t \in [0,2]_{\mathbb{T}}, \tag{4.1}$$

$$u(0) = \frac{1}{2}u\Big(\frac{1}{4}\Big) + \frac{1}{6}u\Big(\frac{3}{4}\Big), \quad u^{\Delta}(2) = 0, \tag{4.2}$$

where

$$f(t,w,v) = \begin{cases} \frac{t}{1000} + \frac{2w^3}{3} + (\frac{v}{100})^3, & w \leq 3, \\ \frac{t}{1000} + 18 + (\frac{v}{100})^3, & w > 3. \end{cases}$$

Clearly, assumptions (H1)–(H5) hold and $f(t,0,0) \not\equiv 0$ on $[0,2]_{\mathbb{T}}$. We choose $r_1 = 1/2$, $r_2 = 140$, $b = 2$, and $l_1 = 1/4$, $l_2 = 80$. So $0 < r_1 < b < 2b < r_2$ and $0 < l_1 < l_2$. By doing some calculations, we obtain

$$\begin{gathered} M = \int_0^{\eta} \phi_q\Big(\int_{\eta}^{T} a(\tau)\nabla\tau\Big)\Delta s = 1, \\ L = \phi_q\Big(\int_0^{T} a(\tau)\nabla\tau\Big) = 2, \end{gathered}$$

and

$$\begin{aligned} N &= \int_0^{T} \phi_q\Big(\int_s^{T} a(\tau)\nabla\tau\Big)\Delta s + \frac{1}{1 - \sum_{i=1}^{m-2}\alpha_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_q\Big(\int_s^{T} a(\tau)\nabla\tau\Big)\Delta s \\ &< \tilde{N} \\ &= \int_0^{T} \phi_q\Big(\int_0^{T} a(\tau)\nabla\tau\Big)\Delta s + \frac{1}{1 - \sum_{i=1}^{m-2}\alpha_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_q\Big(\int_0^{T} a(\tau)\nabla\tau\Big)\Delta s \\ &= \frac{19}{4}. \end{aligned}$$

As a result, $f(t,w,v)$ satisfies

$$f(t,w,v) \le \min\big\{\phi_p\Big(\frac{r_2}{\tilde{N}}\Big), \phi_p\Big(\frac{l_2}{L}\Big)\big\} \approx 29.4736 < \min\big\{\phi_p\Big(\frac{r_2}{N}\Big), \phi_p\Big(\frac{l_2}{L}\Big)\big\},$$

for $0 \le t \le 2$, $0 \le w \le 140$, $|v| \le 80$;

$$f(t,w,v) > \phi_p\big(\frac{b}{M}\big) = 2,$$

for $1 \le t \le 2$, $2 \le w \le 4$, $|v| \le 80$;

$$f(t,w,v) < \min\big\{\phi_p\Big(\frac{r_1}{\tilde{N}}\Big), \phi_p\Big(\frac{l_1}{L}\Big)\big\} \approx 0.1053 < \min\big\{\phi_p\Big(\frac{r_1}{N}\Big), \phi_p\Big(\frac{l_1}{L}\Big)\big\},$$

for $0 \le t \le 2$, $0 \le w \le \frac{1}{2}$, $|v| \le 1/4$. Hence, by Theorem 3.1, BVP (4.1) and (4.2) has at least three non-negative solutions, two of them positive, u_1, u_2, u_3 such that

$$\max_{t\in[0,2]_{\mathbb{T}}}\{u_1(t)\} < \frac{1}{2}, \quad \sup_{t\in[0,2]_{\mathbb{T}}} |u_1^{\Delta}(t)| < \frac{1}{4};$$
$$2 < \min_{t\in[1,2]_{\mathbb{T}}}\{u_2(t)\} \le \max_{t\in[0,2]_{\mathbb{T}}}\{u_2(t)\} \le 140, \quad \sup_{t\in[0,2]_{\mathbb{T}}} |u_2^{\Delta}(t)| \le 80;$$
$$\min_{t\in[1,2]_{\mathbb{T}}}\{u_3(t)\} < 2, \quad \frac{1}{2} < \max_{t\in[0,2]_{\mathbb{T}}}\{u_3(t)\} < 4, \quad \frac{1}{4} < \sup_{t\in[0,2]_{\mathbb{T}}} |u_3^{\Delta}(t)| \le 80.$$

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