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Energy Transfer in Two-Level Quantum Systems via Speed Gradient-Based Algorithm ⋆

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Abstract: We develop the speed gradient-based algorithm for controlled transfer of energy in a two-level quantum system towards a predefined value of energy using as control spectral density of incoherent photons. The algorithm can stabilize energy at a value less than one half of the energy gap between the two system states and is shown to be more effective for cooling than for heating.

Keywords: Quantum system, speed gradient.

1. INTRODUCTION

The processes related to cooling or heating play an important role in quantum engineering where control of energy of a sub-system placed in the reservoir with given properties may serve for effective manipulation with the efficiency of elementary quantum engine. Intensive studies of such processes for quantum particles (especially for cooling) were stimulated by the 1997 Nobel Prize in Physics (Phillips, Chu, Cohen-Tannoudji), see the reviews by Phillips (1998); Metcalf and van der Straten (1999); Balykin et al. (2000), and different approaches were applied by Ketterle and van Druten (1996); Schiller and Lammerzahl (2003), including toy modeling with Brownian particle in Borisenok and Rozhdestvensky (2011).

A great progress has been achieved in the algorithm of so-called phonon-induced dynamic resonance energy transfer (a coherent energy-transfer mechanism for quantum systems), where phonon interactions are able to actually enhance the energy transfer, see Lim et al. (2014).

The important particular system for energy-manipulating control is two-level quantum system (qubit). Some papers describe qubit interacting in uncontrollable way with a quantum environment. The time-optimal control of dissipative spin 1/2 particles was studied by Lapert et al. (2010). The energy minimization problem for two-level dissipative quantum systems was considered in Bonnard et al. (2010). The accent on the qubit decoherence was made by Thorwart et al. (2000) and dephasing in Takane and Murakami (2005) by π-pulses. The laser π-pulse optimal control for 2- and 3-level systems was studied by Boscain et al. (2002). Control of relaxation of a qubit was analyzed by Mukherjee et al. (2013).

In the frame of open-loop control the maximum population transfer was studied in a periodically driven quantum system by Poggi et al. (2014).

Environment can influence the qubit or other controlled system not only as a passive destructive medium but can also serve as an efficient tool for controlling the system, see Pechen and Rabitz (2006); Pechen (2011). In Pechen and Rabitz (2006) environment of incoherent photons was used as control to prepare desired states of multi-level quantum systems. Moreover, combination of incoherent control by engineered reservoir (spectral density of incoherent photons) and coherent control (shaped laser pulses) was shown by Pechen (2011) to be rich enough to realize strongest degree of quantum state control—-complete density matrix controllability of open quantum systems. That is, it was found that one can steer with coherent and incoherent controls any initial density matrix of an n-level quantum system to any target density matrix.

The important case is time optimal control (see for instance Khaneja et al. (2001) for spin systems). Time-optimal theory has been applied by Kallush and Kosloff (2006) to the 2-level system governed by the Liouville equation to protect the system against noise. The population transfer in two-level quantum systems has been also performed with the algorithm of dephasing noise and/or systematic frequency errors by Lu et al. (2013).

In the work of Liu et al. (2005) Markovian feedback of the white-noise measurement record via a Hamiltonian was considered, and this model was applied to stabilize the state of two-level quantum system. Another approach of feedback is to construct the system of dynamical equations related to the qubit state, which includes the control field, and then to construct a reasonable algorithm that allows to find the control field via the dynamical variables. In this approach one does not need to concentrate on the details of real-time monitoring and driving the system, like

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in Liu et al. (2005), Lloyd (2000), because the final shape of the control field can be accepted as the theoretically calculated recommended form to achieve the control goal. The simple way to do it is to define a goal function expressing the desired qubit state, and to demand that the control field must minimize the goal function. This idea leads to speed gradient method of Fradkov (2007) which was applied to control quantum state of a two-level system by Saifullah (2008); Borisenok et al. (2010) in the frame of semi-classical model where the quantum particle was controlled with a classical external field.

In this paper we develop a speed gradient approach for the efficient manipulation with the energy of a two-level quantum system immersed in a quantum reservoir of incoherent photons whose spectral density is used as driving control force. After the description of the quantum model in Section 2, in Section 3 we derive a speed gradient-based algorithm for cooling or heating the system. Section 4 contains analysis of stationary states and stability of the algorithm, followed by numerical simulations for various cases of cooling and heating. We finalize our results and conclusions in Conclusions Section 5.

2. FORMULATION

We consider two-level open quantum system which interacts with reservoir of incoherent photons. Spectral density of incoherent photons is used as control. To set up the control problem, we need to specify the set of states of the controlled system, the space of controls, evolution equation for the system under the action of controls, and objective function.

We denote ground and excited states of the qubit as $|1\rangle$ and $|2\rangle$, correspondingly. Let P_1 and P_2 be projectors on the ground and excited states. Transition frequency between the two states is denoted by ω_0. Free system Hamiltonian is $H_0 = \omega_0 P_2$.

Set of states for a two-level quantum system is the set of all density matrices

$$\mathcal{D}_2 = \{\rho \in \mathcal{M}_2 := \mathbb{C}^{2\times 2} \,|\, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

This set of all density matrices is the most general set of states of a two-level quantum system.

We consider control by spectral density of incoherent radiation which interacts with the system as was proposed by Pechen and Rabitz (2006); Pechen (2012). Hence control is a non-negative function $n(t) = n_{12}(t) \geq 0$ which represents spectral density of incoherent photons at the transition frequency ω_0.

Evolution of the system density matrix is Markovian with master equation of Kossakowski-Gorini-Sudarschan-Lindblad form

$$\frac{d\rho_t}{dt} = -i[H_0 + u(t)V, \rho_t] + \gamma\left[(n(t)+1)L_{12}(\rho_t) + n(t)L_{21}(\rho_t)\right] \quad (1)$$

where $\gamma = \gamma_{12} > 0$ is the Einstein coefficient for transition between states $|2\rangle$ and $|1\rangle$, and

$$L_{12}(\rho) = 2\rho_{22}P_1 - P_2\rho - \rho P_2$$

$$L_{21}(\rho) = 2\rho_{11}P_2 - P_1\rho - \rho P_1$$

Here $\rho_{11} =< 1|\rho|1 >$ and $\rho_{22} =< 2|\rho|2 >$.

In this work we assume that coherent control is switched off, so that $u(t) = 0$. We consider cooling or heating the system towards a predefined value of energy E_0. Average energy of the system at time t is $\bar{E} = \mathrm{Tr}(H_0\rho_t) = \omega_0\rho_{22}$, where $\rho_{22} = \langle 2|\rho_t 2|\rangle$ is the population of the excited level. It is easy to check that

$$\dot{\bar{E}} = 2\gamma\omega_0[n\rho_{11} - (n+1)\rho_{22}] = 2\gamma[n\omega_0 - (2n+1)\bar{E}] \quad (2)$$

where we use $\mathrm{Tr}\rho = \rho_{11} + \rho_{22} = 1$.

The final step is to set up the goal of control which is formulated as minimization of (non-negative) goal function

$$Q_t = \frac{1}{2}(\bar{E} - E_0)^2, \qquad Q_t \to 0 \text{ as } t \to \infty$$

3. SPEED GRADIENT-BASED ALGORITHM FOR COOLING OR HEATING THE SYSTEM

Speed gradient-based algorithm is based on the gradient of speed of changing the goal function along a trajectory of the system, see Fradkov (1979). Speed of changing the goal function Q_t along a trajectory has the form

$$\omega(n) = \partial_{\bar{E}} Q \cdot \dot{\bar{E}} = 2\gamma(\bar{E} - E_0)[n\omega_0 - (2n+1)\bar{E}] \quad (3)$$

Its gradient with respect to controls is

$$\nabla_n \omega(n) = 2\gamma(\bar{E} - E_0)(\omega_0 - 2\bar{E}) \quad (4)$$

General form of speed gradient algorithm is defined by

$$\frac{d}{dt}[n + \psi(n)] = -\Gamma\nabla_n\omega(n) \quad (5)$$

Here $\Gamma \geq 0$ and vector ψ satisfies the pseudogradient condition $\langle\psi, \nabla_n\omega\rangle \geq 0$.

Differential form of the speed gradient algorithm is

$$\frac{dn}{dt} = 4\gamma\Gamma(\bar{E} - E_0)(\bar{E} - \omega_0/2) \quad (6)$$

Finite (linear) form of the speed gradient algorithm is

$$n = 4\gamma\Gamma(\bar{E} - E_0)(\bar{E} - \omega_0/2) \quad (7)$$

For finite form the condition $n \geq 0$ implies

$$\bar{E} \geq \max(E_0, \omega_0/2) \text{ or } \bar{E} \leq \min(E_0, \omega_0/2)$$

Therefore if $E_0 \geq \omega_0/2$, then algorithm may work if $\bar{E} \geq E_0$ that corresponds to cooling of the system to energy E_0. The algorithm leads to negative density and hence does not work if $\bar{E} < E_0$ (when $E_0 \geq \omega_0/2$). It is natural since it is impossible to create inversion of population in a two level system by only incoherent control.

In opposite, if target energy $E_0 \leq \omega_0/2$ then algorithm works if $\bar{E} \leq E_0$ that corresponds to heating of the system to energy E_0.

We introduce dimensionless variable $x = \bar{E}/E_0$ and define $r = \omega_0/(2E_0)$, $\tilde{\Gamma} = 4\gamma\Gamma E_0^2$. Restriction on the range of r comes from the fact that it is meaningless for the target energy E_0 to be greater that ω_0. Therefore we have $E_0 \leq \omega_0$ that implies $r \geq 1/2$. Energy $\bar{E}$ of the system at any time is non-negative and can not exceed ω_0, $0 \leq \bar{E} \leq \omega_0$. It implies that x should satisfy the restriction $0 \leq x \leq 2r$.

In these dimensionless variabless for the finite form of speed gradient algorithm by (7) we get:

$$n = \tilde{\Gamma}(x-1)(x-r) \quad (8)$$

$$\dot{x} = 4\gamma\left[nr - \left(n + \frac{1}{2}\right)x\right] \quad (9)$$

In these units $r > 1$, $0 \le x < 1$ stands for heating, and $r < 1$, $1 < x \le 2r$ for cooling.

Substitution of the control (8) in the dynamical equation (9) gives

$$\dot{x} = -4\gamma\left[\tilde{\Gamma}(x-1)(x-r)^2 + \frac{x}{2}\right] \quad (10)$$

For the differential form (6), correspondingly, we get:

$$\frac{dn}{dt} = \tilde{\Gamma}(x-1)(x-r)\ . \quad (11)$$

Thus, the method of speed gradient in the finite form is defined by Eqs. (8) and (9) which together produce (10). Speed gradient method in the differential form is defined by Eqs. (9) and (11).

4. ANALYSIS AND SIMULATIONS

4.1 Stationary states for the finite speed gradient algorithm

Stationary states of (10) are solutions to the cubic equation

$$\tilde{\Gamma}(x_* - 1)(x_* - r)^2 + \frac{x_*}{2} = 0$$

This equation has only one real solution:

$$x_* = \frac{C}{6\tilde{\Gamma}} + \frac{2\tilde{\Gamma}(r-1)^2 - 3}{3C} + \frac{2r+1}{3}\ , \quad (12)$$

where

$$A = -18(2r+1) - 8\tilde{\Gamma}(r-1)^3\ ;$$
$$B = 8r(r-1)^3\tilde{\Gamma} + 8r^2 + 20r - 1 + \frac{2}{\tilde{\Gamma}}\ ; \quad (13)$$
$$C = \tilde{\Gamma}^{2/3}\left(A + 6\sqrt{3B}\right)^{1/3}\ .$$

By $B > 0$ we get:

$$(r-1)^3\left(\tilde{\Gamma} - \Gamma_+\right)\left(\tilde{\Gamma} - \Gamma_-\right) > 0 \quad (14)$$

with

$$\Gamma_\pm = \frac{-(8r^2 + 20r - 1) \pm (8r+1)^{3/2}}{16r(r-1)^3}\ . \quad (15)$$

Here we used the property $r > 0$.

The typical behavior for the solution (12) is presented on Fig. 1.

4.2 Remarks on the stabilization

In the δ-neighborhood of the control goal $x = 1 + \delta$ ($|\delta| \ll 1$), where $\delta < 0$ corresponds to heating, and $\delta > 0$ to cooling, by (10) we get:

$$\dot{\delta} \simeq -4\gamma\left[\tilde{\Gamma}(1-r)^2\delta + \frac{1+\delta}{2}\right] < 0\ , \quad (16)$$

with the solution:

$$\delta(t) = -\frac{1}{G} + C \cdot \exp\{-2\gamma G t\}\ , \quad (17)$$

with the positive constant $G = 1 + 2\tilde{\Gamma}(1-r)^2$, that provides the stability for our algorithm.

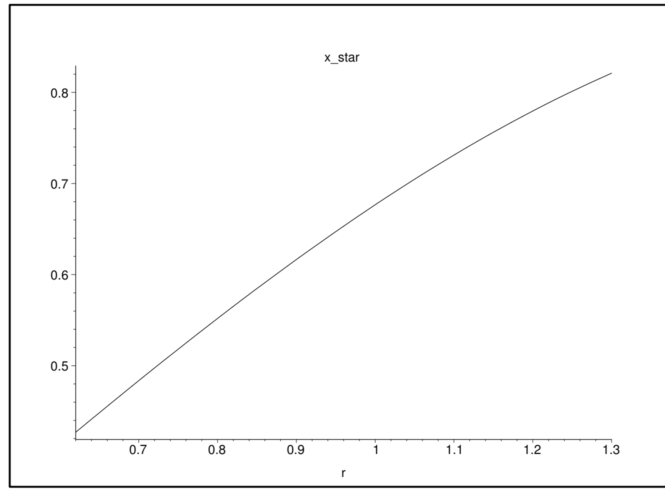

Fig. 1. The function x_* vs r, $\tilde{\Gamma} = 10$.

Correspondingly, by (8),

$$n \simeq \tilde{\Gamma}(1-r)\delta\ . \quad (18)$$

If $\tilde{\Gamma} \gg 1$, the asymptotic as $t \to \infty$ is given by $\delta \simeq 1/2\tilde{\Gamma}(1-r)^2$ and $n \simeq 1/2(r-1)$. In this case the requirement $n \ge 0$ implies $r > 1$ that corresponds to $E_0 < \omega_0/2$. This is in agreement that in a two level system one can excite not more that 1/2 population to upper level (it is impossible to create population inversion in a two level system by means of only incoherent sources). Therefore it is impossible to stabilize energy at value more than $\omega_0/2$.

4.3 Numerical simulations

The results of numerical simulations for the finite form of speed gradient algorithm are given on Fig. 2 for heating ($\tilde{\Gamma} = 5, 10, 15$, $\gamma = 2$, $r = 2$, with the initial condition $x(0) = 0.2$) and on Fig. 3 for cooling ($\tilde{\Gamma} = 5, 10, 15$, $\gamma = 2$, $r = 2/3$, with the initial condition $x(0) = 1.3 < 2r$).

The results of numerical simulations for the differential form are presented on Fig. 3 for heating and on Fig. 4 for cooling for the same values of the parameters (solid lines). The value of $\tilde{\Gamma}$ here defines the characteristic time scale $1/\tilde{\Gamma}$ of the asymptotic achievement of the control goal.

Figures 4–5 demonstrate that in the frame of our algorithm the differential form of speed gradient approach works well only for heating. Such effect is well-known in the theory of gradient feedback, see, for instance, Andrievski et al. (1996). The corresponding control fields $n(t)$ are given by (11) and are plotted on the same Figs. as pointed lines.

We can see from Fig. 4–5 that the control field for cooling cannot be stabilized with the differential algorithm because of the structure of Eqs. (10) and (11), while for heating the control signal tends to a constant as $t \to \infty$.

5. CONCLUSIONS

This work shows that speed gradient algorithm in its quantum formulation is a successful tool for manipulating the energy of a two-level quantum system. The algorithm

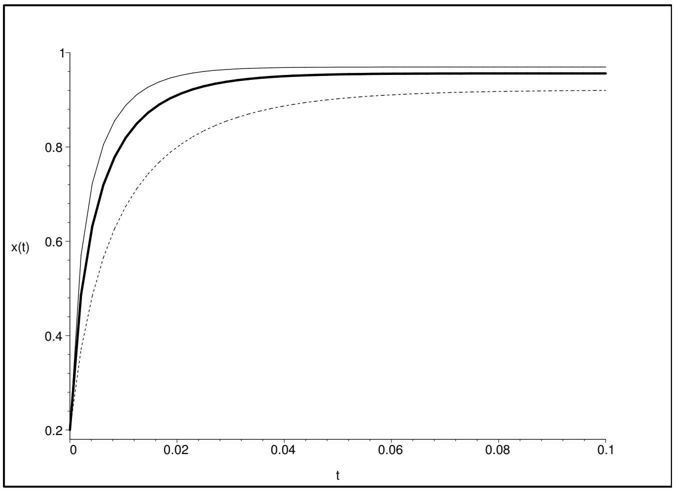

Fig. 2. Heating with the finite form of speed gradient algorithm for $\tilde{\Gamma} = 5$ (solid thin line), $\tilde{\Gamma} = 10$ (solid thick line), and $\tilde{\Gamma} = 15$ (dashdot line). Other parameters are $\gamma = 2$, $r = 2$, the initial condition is $x(0) = 0.2$.

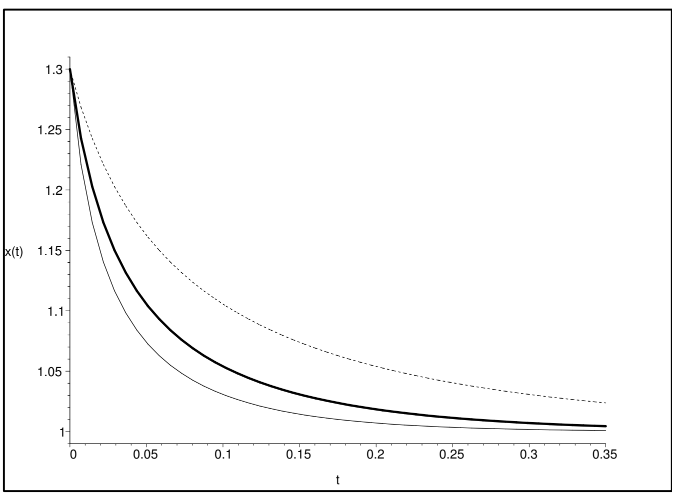

Fig. 3. Cooling with the finite form of speed gradient algorithm for $\tilde{\Gamma} = 5$ (solid thin line), $\tilde{\Gamma} = 10$ (solid thick line), and $\tilde{\Gamma} = 15$ (dashdot line). Other parameters are $\gamma = 2$, $r = 2/3$, the initial condition is $x(0) = 1.3$.

is more effective for cooling such a system rather then for heating. This approach can be extended to multi-level systems and allows to consider different scenarios of energy transfer between the quantum system and the environment.

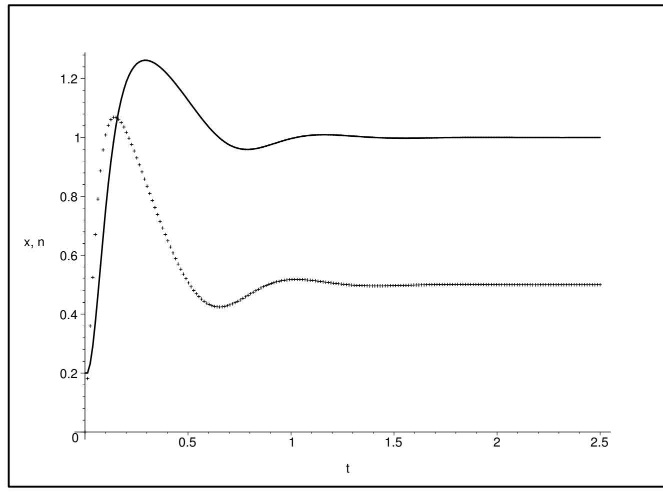

Fig. 4. Heating with the differential form of speed gradient algorithm. $\tilde{\Gamma} = 10$, $\gamma = 2$, $r = 2$, with the initial condition $x(0) = 0.2$. Solid line: $x(t)$; Pointed line: $n(t)$.

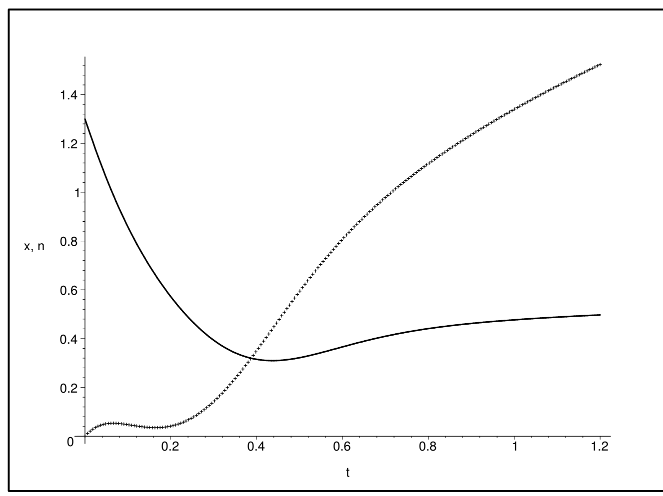

Fig. 5. Cooling with the differential form of speed gradient algorithm. $\tilde{\Gamma} = 10$, $\gamma = 2$, $r = 2/3$, with the initial condition $x(0) = 1.3$. Solid line: $x(t)$; Pointed line: $n(t)$.

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