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ŞUAYİP YÜZBAŞI

MEHMET EMİN TAMAR

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A new approaching method for linear neutral delay differential equations by using Clique polynomials

Şuayip YÜZBAŞI[1,*], **Mehmet Emin TAMAR**[1,2]
[1]Department of Mathematics, Faculty of Science, Bartın University, Bartın, Turkiye
[2]Department of Engineering Sciences, Faculty of Engineering, Abdullah Gül University, Kayseri, Turkiye

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Abstract: This article presents an efficient method for obtaining approximations for the solutions of linear neutral delay differential equations. This numerical matrix method, based on collocation points, begins by approximating $y'(u)$ using a truncated series expansion of Clique polynomials. This method is constructed using some basic matrix relations, integral operations, and collocation points. Through this method, the neutral delay problem is transformed into a system of linear algebraic equations. The solution of this algebraic system determines the coefficients of the approximate solution obtained by this method. The efficiency, accuracy, and error analysis of this method are demonstrated by applying it to several numerical problems. All calculations in this method have been performed using the computer program MATLAB R2021a.

Key words: Neutral delay differential equations, collocation method, Clique polynomials, approximate solutions

1. Introduction

Neutral delay differential equations hold significant importance in various scientific areas. Many problems in these domains can be addressed by modeling them with neutral delay differential equations. Applications can be found in mechanics, economics, biology, electrodynamics, and more. [1, 6, 9, 11, 17, 18, 31, 32].

Some of the methods applied to solve delay differential equations are [4, 5, 7, 8, 15, 16, 19, 20, 24, 30, 35, 42, 45]. Additionally, various matrix methods have been used, employing Taylor, Legendre, Bessel, and Legendre polynomials to obtain approximate solutions for these types of problems [26, 29, 37–41, 49, 52, 53]. Moreover, a recent matrix method utilizing Clique polynomials for approximate solutions of coupled differential equations systems can be observed in [28].

Using the aforementioned information, we present a new method that employs Clique polynomials to approximate solutions for the linear neutral delay differential equation defined in [27] as

$$y'(u) = H(u)y(u) + \sum_{i=1}^{J} P_i(u)y(\lambda_i u) + \sum_{j=1}^{K} Q_j(u)y'(\mu_j u) + g(u), \quad 0 \le a \le u \le b \tag{1}$$

with the initial condition

*Correspondence: suayipyuzbasi@bartin.edu.tr
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$$y(a) = \gamma. \tag{2}$$

Here, $y(u)$ is an unknown function and $H(u)$, $P_i(u)$, $Q_j(u)$ and $g(u)$ are the known functions that are defined on $0 \leq a \leq u \leq b$ and λ_i, μ_j and γ are the constants.

This work introduces a new method based on Clique polynomials, as introduced in [21, 22], for the problem (1)-(2).

The method starts by assuming that $y'(u)$ in the problem has a series expansion based on Clique polynomials, given by

$$y'(u) = \sum_{i=0}^{M} a_i h(u, K_i), \tag{3}$$

where a_i, for $i = 0, 1, 2, \ldots, M$, are the coefficients of Clique polynomials to be determined. Clique polynomials are defined as

$$h(u, K_i) = \sum_{j=0}^{i} \binom{i}{j} u^j$$

for the complete graph K_i with i vertices, where $i = 0, 1, 2, \ldots, M$, and $h(u, K_0) = 1$.

For example, for $i = 1, 2, 3$, some expansions of Clique polynomials are:

$$h(u, K_1) = 1 + u,$$

$$h(u, K_2) = 1 + u + u^2,$$

$$h(u, K_3) = 1 + u + u^2 + u^3.$$

2. Basic matrix relations

In this section, the Clique polynomial approximation of $y'(u)$ is expressed in matrix form using some basic matrix relations, as commonly used in many articles [2, 28, 37–39]. The approximate solution $y(u)$ is obtained by employing some integral operations instead of the derivative operations typically used in most articles. Now, let us explore these basic matrix relations.

First, let $C(u) = h(u, K_i)$, for $i = 0, 1, 2, \ldots, M$. Clique polynomials can be expressed in matrix form as

$$\mathbf{C}^T(u) = \mathbf{D}\mathbf{U}^T(u) \Leftrightarrow \mathbf{C}(u) = \mathbf{U}(u)\mathbf{D}^T \tag{4}$$

with

$$\mathbf{C}(u) = \begin{bmatrix} h(u, K_0) & h(u, K_1) & h(u, K_2) & \cdots & h(u, K_M) \end{bmatrix},$$
$$\mathbf{U}(u) = \begin{bmatrix} 1 & u & u^2 & \cdots & u^M \end{bmatrix},$$

and

$$\mathbf{D} = \begin{bmatrix} \binom{M}{0} & 0 & 0 & \cdots & 0 \\ \binom{M}{0} & \binom{M}{1} & 0 & \cdots & 0 \\ \binom{M}{0} & \binom{M}{1} & \binom{M}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{M}{0} & \binom{M}{1} & \binom{M}{2} & \cdots & \binom{M}{M} \end{bmatrix}.$$

Now, we can write the relation (3) in the matrix form as

$$y'(u) = \mathbf{U}(u)\,\mathbf{D}^T\mathbf{A} \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_M \end{bmatrix}^T.$$

Integrating equation (5) from a to u, we get

$$\begin{aligned} y(u) - y(a) &= \int_a^u \sum_{i=0}^{M} a_i h(\xi, K_i)\, d\xi = \int_a^u \mathbf{U}(\xi)\,\mathbf{D}^T\mathbf{A} d\xi \Rightarrow \\ y(u) &= y(a) + \left(\int_a^u \begin{bmatrix} 1 & \xi & \xi^2 & \cdots & \xi^M \end{bmatrix} d\xi \right) \mathbf{D}^T\mathbf{A} \Rightarrow \\ y(u) &= y(a) + \begin{bmatrix} (u-a) & \frac{u^2}{2} - \frac{a^2}{2} & \frac{u^3}{2} - \frac{a^3}{2} & \cdots & \frac{u^{M+1}}{2} - \frac{a^{M+1}}{2} \end{bmatrix} \mathbf{D}^T\mathbf{A} \\ &= y(a) + \left[\tilde{\mathbf{U}}(u) - \tilde{\mathbf{U}}(a) \right] \mathbf{D}^T\mathbf{A} \end{aligned} \tag{6}$$

where

$$\tilde{\mathbf{U}}(u) = \begin{bmatrix} u & \frac{u^2}{2} & \frac{u^3}{3} & \cdots & \frac{u^{M+1}}{M+1} \end{bmatrix}.$$

Using the condition (2) in equation (6), we can obtain

$$y(u) = \gamma + \left[\tilde{\mathbf{U}}(u) - \tilde{\mathbf{U}}(a) \right] \mathbf{D}^T\mathbf{A}. \tag{7}$$

The expression $\tilde{\mathbf{U}}(u)$ in equation (7) can be expressed in the form

$$\begin{aligned} \tilde{\mathbf{U}}(u) &= \begin{bmatrix} u & \frac{u^2}{2} & \frac{u^3}{3} & \cdots & \frac{u^{M+1}}{M+1} \end{bmatrix} \\ &= u\mathbf{U}(u)\,\mathbf{M} \end{aligned} \tag{8}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{M+1} \end{bmatrix}.$$

Hence, if equation (8) is written in equation (6), then it gives

$$y(u) = \gamma + \left[u\mathbf{U}(u)\,\mathbf{M} - a\mathbf{U}(a)\,\mathbf{M} \right] \mathbf{D}^T\mathbf{A}. \tag{9}$$

Substituting $u \to \lambda_i u$ into equation (9), we obtain

$$y(\lambda_i u) = \gamma + [\lambda_i u \mathbf{U}(\lambda_i u)\mathbf{M} - a\mathbf{U}(a)\mathbf{M}]\mathbf{D}^T\mathbf{A}. \tag{10}$$

The relation between $\mathbf{U}(u)$ and $\mathbf{U}(\lambda_i u)$ can be given by

$$\mathbf{U}(\lambda_i u) = \mathbf{U}(u)\mathbf{B}(\lambda_i) \tag{11}$$

where

$$\mathbf{B}(\lambda_i) = \begin{bmatrix} \lambda_i^0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i^1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i^M \end{bmatrix}.$$

Hence, equation (10) can be written as

$$y(\lambda_i u) = \gamma + [\lambda_i u \mathbf{U}(u)\mathbf{B}(\lambda_i)\mathbf{M} - a\mathbf{U}(a)\mathbf{M}]\mathbf{D}^T\mathbf{A}. \tag{12}$$

Putting $u \to \mu_j u$ in equation (5), we can write

$$\begin{aligned} y'(\mu_i u) &= \mathbf{U}(\mu_i u)\mathbf{D}^T\mathbf{A} \\ &= \mathbf{U}(u)\mathbf{B}(\mu_j)\mathbf{D}^T\mathbf{A}. \end{aligned} \tag{13}$$

Here, $\mathbf{B}(\mu_j)$ can be seen from $\mathbf{B}(\lambda_i)$ in equation (11).

3. Method of solution

In this section, the method which gives the approximate solution $y(u)$ of the problem (1)-(2) is given by using the matrix relations introduced in Section 2.

Now substituting equations (5), (9), (10), (12) and (13) into equation (1), we deduce the basic matrix relation of equation (1) as

$$\begin{aligned} \mathbf{U}(u)\mathbf{D}^T\mathbf{A} = H(u)&\left(\gamma + [u\mathbf{U}(u)\mathbf{M} - a\mathbf{U}(a)\mathbf{M}]\mathbf{D}^T\mathbf{A}\right) \\ &+ \sum_{i=1}^{J} P_i(u)\left[\gamma + [\lambda_i u\mathbf{U}(u)\mathbf{B}(\lambda_i)\mathbf{M} - a\mathbf{U}(a)\mathbf{M}]\mathbf{D}^T\mathbf{A}\right] \\ &+ \sum_{j=1}^{K} Q_j(u)\mathbf{U}(u)\mathbf{B}(\mu_j)\mathbf{D}^T\mathbf{A} + g(u) \end{aligned}$$

where $0 \le a \le u \le b$. Rearranging this expression, we get

$$\left\{ \begin{array}{c} \mathbf{U}(u)\mathbf{D}^T - H(u)[u\mathbf{U}(u) - a\mathbf{U}(a)]\mathbf{M}\mathbf{D}^T \\ -\sum\limits_{i=1}^{J} P_i(u)[\lambda_i u\mathbf{U}(u)\mathbf{B}(\lambda_i) - a\mathbf{U}(a)]\mathbf{M}\mathbf{D}^T \\ -\sum\limits_{j=1}^{K} Q_j(u)\mathbf{U}(u)\mathbf{B}(\mu_j)\mathbf{D}^T \end{array} \right\}\mathbf{A}$$

$$= g(u) + \gamma H(u) + \sum_{i=1}^{J} \gamma P_i(u). \tag{14}$$

Let us define the collocation points as

$$u_s = a + \frac{b-a}{M}s,\ s = 0,\ 1,\ 2,\ \ldots,\ M.$$

Writing $u \to u_s$ in equation (14), we have

$$\left\{ \begin{array}{c} \mathbf{U}(u_s)\,\mathbf{D}^T - H(u_s)\left[u_s\mathbf{U}(u_s) - a\mathbf{U}(a)\right]\mathbf{MD}^T \\ -\sum\limits_{i=1}^{J} P_i(u_s)\left[\lambda_i u_s \mathbf{U}(u_s)\,\mathbf{B}(\lambda_i) - a\mathbf{U}(a)\right]\mathbf{MD}^T \\ -\sum\limits_{j=1}^{K} Q_j(u_s)\,\mathbf{U}(u_s)\,\mathbf{B}(\mu_j)\,\mathbf{D}^T \end{array} \right\}\mathbf{A}$$

$$= g(u_s) + \gamma H(u_s) + \sum_{i=1}^{J} \gamma P_i(u_s)\,. \tag{15}$$

The system (15) can be written in the matrix form

$$\left\{ \begin{array}{c} \mathbf{UD}^T - \left[\mathbf{H}\bar{\mathbf{U}}\mathbf{U} - a\bar{\mathbf{H}}\mathbf{U}_a\right]\mathbf{MD}^T \\ -\sum\limits_{i=1}^{J}\left[\lambda_i \mathbf{P}_i\bar{\mathbf{U}}\mathbf{UB}(\lambda_i) - a\bar{\mathbf{P}}_i\mathbf{U}_a\right]\mathbf{MD}^T \\ -\sum\limits_{j=1}^{K}\mathbf{Q}_j\mathbf{UB}(\mu_j)\,\mathbf{D}^T \end{array} \right\}\mathbf{A}$$

$$= \mathbf{G} + \gamma\bar{\mathbf{H}} + \gamma\sum_{i=1}^{J}\bar{\mathbf{P}}_i \tag{16}$$

where

$$\mathbf{H} = \begin{bmatrix} H(u_0) & 0 & 0 & \cdots & 0 \\ 0 & H(u_1) & 0 & \cdots & 0 \\ 0 & 0 & H(u_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H(u_M) \end{bmatrix},\ \bar{\mathbf{H}} = \begin{bmatrix} H(u_0) \\ H(u_1) \\ H(u_2) \\ \vdots \\ H(u_M) \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}(u_0) \\ \mathbf{U}(u_1) \\ \mathbf{U}(u_2) \\ \vdots \\ \mathbf{U}(u_M) \end{bmatrix},\ \mathbf{U}_a = \begin{bmatrix} \mathbf{U}(a) \\ \mathbf{U}(a) \\ \mathbf{U}(a) \\ \vdots \\ \mathbf{U}(a) \end{bmatrix},\ \bar{\mathbf{U}} = \begin{bmatrix} u_0 & 0 & 0 & \cdots & 0 \\ 0 & u_1 & 0 & \cdots & 0 \\ 0 & 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_M \end{bmatrix},$$

$$\mathbf{P}_i = \begin{bmatrix} P_i(u_0) & 0 & 0 & \cdots & 0 \\ 0 & P_i(u_1) & 0 & \cdots & 0 \\ 0 & 0 & P_i(u_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_i(u_M) \end{bmatrix}, \; \bar{\mathbf{P}}_i = \begin{bmatrix} P_i(u_0) \\ P_i(u_1) \\ P_i(u_2) \\ \vdots \\ P_i(u_M) \end{bmatrix},$$

$$\mathbf{Q}_j = \begin{bmatrix} Q_j(u_0) & 0 & 0 & \cdots & 0 \\ 0 & Q_j(u_1) & 0 & \cdots & 0 \\ 0 & 0 & Q_j(u_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q_j(u_M) \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} g(u_0) \\ g(u_1) \\ g(u_2) \\ \vdots \\ g(u_M) \end{bmatrix}.$$

Therefore, the basic matrix equation (15) of equation (1) can be expressed as

$$\mathbf{WA} = \mathbf{F} \text{ or } [\mathbf{W}; \mathbf{F}] \tag{17}$$

where

$$\mathbf{W} = \mathbf{UD}^T - \left[\mathbf{H}\bar{\mathbf{U}}\mathbf{U} - a\bar{\mathbf{H}}\mathbf{U}_a\right]\mathbf{MD}^T - \sum_{i=1}^{J}\left[\lambda_i \mathbf{P}_i \bar{\mathbf{U}}\mathbf{UB}(\lambda_i) - a\bar{\mathbf{P}}_i\mathbf{U}_a\right]\mathbf{MD}^T$$

$$-\sum_{j=1}^{K}\mathbf{Q}_j\mathbf{UB}(\mu_j)\mathbf{D}^T$$

and

$$\mathbf{F} = \mathbf{G} + \gamma\bar{\mathbf{H}} + \gamma\sum_{i=1}^{J}\bar{\mathbf{P}}_i.$$

Finally, the matrix $\mathbf{A}$, whose entries are unknown coefficients, can be found by solving the linear algebraic system (16). Then, substituting the determined matrix $\mathbf{A}$ into equation (7), we deduce the approximate solution as

$$y_M(u) = \gamma + \left[\tilde{\mathbf{U}}(u) - \tilde{\mathbf{U}}(a)\right]\mathbf{D}^T\mathbf{A}. \tag{18}$$

4. Error estimation

In this section, an error estimation is provided for the Clique polynomial solution (18) by utilizing the residual error function [33]. Subsequently, an improved Clique polynomial solution (18) is obtained. Let us now consider the following theorem for the error estimation.

Theorem 1 *Assume that $y(u)$ is the exact solution and $y_M(u)$ is the approximate solution of the present method with M-th degree for the problem (1)-(2). Then the error problem can be expressed as*

$$e'_M(u) = H(u)\, e_M(u) + \sum_{i=1}^{J} P_i(u)\, e_M(\lambda_i u) + \sum_{j=1}^{K} Q_j(u)\, e'_M(\mu_j u) - R_M(u) \tag{19}$$

with the initial condition

$$e_M(a) = 0 \tag{20}$$

where $e_M(u) = y(u) - y_M(u)$ and $R_M(u)$ is the residual function of the problem (1)-(2).

Proof Putting the approximate solution (18) in equation (1), we have the residue function as

$$R_M(u) = y'_M(u) - H(u)\, y_M(u) - \sum_{i=1}^{J} P_i(u)\, y_M(\lambda_i u) - \sum_{j=1}^{K} Q_j(u)\, y'_M(\mu_j u) - g(u)$$

or we can write

$$y'_M(u) = H(u)\, y_M(u) + \sum_{i=1}^{J} P_i(u)\, y_M(\lambda_i u) + \sum_{j=1}^{K} Q_j(u)\, y'_M(\mu_j u) + g(u) + R_M(u). \tag{21}$$

Subtracting equation (21) from equation (1) side-by-side, we get

$$y'(u) - y'_M(u) = H(u)\,(y(u) - y_M(u)) + \sum_{i=1}^{J} P_i(u)\,(y(\lambda_i u) - y_M(\lambda_i u))$$

$$+ \sum_{j=1}^{K} Q_j(u)\,(y'(\mu_j u) - y'_M(\mu_j u)) - R_M(u). \tag{22}$$

If we write the error function as

$$y(u) - y_M(u) = e_M(u)$$

then we have

$$y'(u) - y'_M(u) = e'_M(u).$$

Using these error functions, we can write equation (22) as

$$e'_M(u) = H(u)\, e_M(u) + \sum_{i=1}^{J} P_i(u)\, e_M(\lambda_i u) + \sum_{j=1}^{K} Q_j(u)\, e'_M(\mu_j u) - R_M(u). \tag{23}$$

On the other hand, the approximate solution (18) satisfies the condition (2). So we have the condition

$$e_M(a) = y(a) - y_M(a) = 0. \tag{24}$$

Thus, equations (23) and (24) give the required result. □

Using the same method in Section 3, by taking the truncation boundary as L instead of M where $L > M$, we begin the method with

$$e'_{M,L}(u) = \sum_{i=0}^{L} a_i h(u, K_i).$$

After the application of the method, we find the approximate solution as

$$e_{M,L}(u) = \left[\tilde{\mathbf{U}}(u) - \tilde{\mathbf{U}}(a)\right]\mathbf{D}^T\mathbf{A}.$$

Corollary 1.1 *The function $e_{M,L}(u)$ is the estimated error function, which is an approximation for the actual error function $e_M(u)$.*

Corollary 1.2 *We can obtain a better approximate solution, called the improved approximate solution $y_{M,L}(u)$, by adding the approximate solution $y_M(u)$ and the estimated error function $e_{M,L}(u)$, i.e., $y_{M,L}(u) = y_M(u) + e_{M,L}(u)$.*

Corollary 1.3 *If we subtract the improved approximate solution $y_{M,L}(u)$ from the approximate solution $y_M(u)$, we obtain the improved error function as $E_{M,L}(u) = y(u) - y_{M,L}(u)$.*

5. Numerical examples

In this section, the method presented in Section 3 is applied to some numerical examples. Additionally, graphs and tables illustrating approximate solutions, comparisons of different methods, and error analysis as shown in Section 4 are provided. All calculations, graphs, and tables were generated using MATLAB R2021a.

Example 5.1 [53] *Let us apply the present method to the following linear neutral delay differential equation:*

$$y'(u) = y(u) - \sin(0.2u)y'(0.25u) + \cos(0.25u)y(0.2u) + \cos(u) - \sin(u), \quad 0 \leq u \leq 1 \tag{25}$$

with the initial condition

$$y(0) = 0.$$

The exact solution of this problem is $y(u) = \sin(u)$. Let the approximation for $y'(u)$ by the truncated series of Clique polynomials be

$$h(u, K_M) = \sum_{j=0}^{3} \binom{3}{j} u^j.$$

Comparing the equations of this example with equations (1) and (2), it can be easily seen that $H(u) = 1$, $P_1(u) = \cos(0.25u)$, $\lambda_1 = 0.2$, $Q_1(u) = \sin(0.2u)$, $\mu_1 = 0.25$, $g(u) = \cos(u) - \sin(u)$, $a = 0$, $\gamma = 0$, and $M = 3$. For $M = 3$, the set of collocation points is obtained as

$$\left\{u_0 = 0, \quad u_1 = 1/3, \quad u_2 = 2/3, \quad u_3 = 1\right\}.$$

Also, using equation (16), we obtain the fundamental matrix equation as

$$\left\{\mathbf{U}\mathbf{D}^T - \mathbf{H}\bar{\mathbf{U}}\mathbf{U}\mathbf{M}\mathbf{D}^T - \lambda_1\mathbf{P}_1\bar{\mathbf{U}}\mathbf{U}\mathbf{B}(\lambda_1)\mathbf{M}\mathbf{D}^T - \mathbf{Q}_1\mathbf{U}\mathbf{B}(\mu_1)\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G}$$

where

$$\mathbf{H}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \bar{\mathbf{H}}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},$$

$$\mathbf{U}=\begin{bmatrix}1&0&0&0\\1&1/3&1/9&1/27\\1&2/3&4/9&8/27\\1&1&1&1\end{bmatrix},\ \bar{\mathbf{U}}=\begin{bmatrix}0&0&0&0\\0&1/3&0&0\\0&0&2/3&0\\0&0&0&1\end{bmatrix},$$

$$\mathbf{P}_1=\begin{bmatrix}1&0&0&0\\0&1723/1729&0&0\\0&0&427/433&0\\0&0&0&187/193\end{bmatrix},\ \bar{\mathbf{P}}_1=\begin{bmatrix}1\\1723/1729\\427/433\\427/433\end{bmatrix},$$

$$\mathbf{Q}_1=\begin{bmatrix}0&0&0&0\\0&-90/1351&0&0\\0&0&-247/1858&0\\0&0&0&-209/1052\end{bmatrix},\mathbf{G}=\begin{bmatrix}1\\1419/2297\\361/2155\\-1005/3337\end{bmatrix}.$$

Then we have the augmented matrix as

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}1&1&1&1&;&1\\1221/1831&2741/1815&620/343&1081/587&;&1419/2297\\384/1147&1710/1001&1523/553&1975/658&;&361/2155\\84/17189&3608/2261&1780/491&2027/463&;&-1005/3337\end{bmatrix}.$$

If this system is solved, then the coefficients of Clique polynomial solution can be found as

$$\mathbf{A}=\begin{Bmatrix}322/323\\662/3557\\-84/319\\303/3773\end{Bmatrix}.$$

Thus, the approximate solution of this problem for $M=3$ can be obtained as

$$y_3(u)=(1.0000e+00)\,u+(4.6444e-003)\,u^2-(1.8302e-001)\,u^3 + (2.0077e-002)\,u^4.$$

By the same operations, we find the approximate solutions for $M=5$ and $M=8$ respectively as

$$y_5(u)=(1.0000e+000)\,u-(2.5817e-005)\,u^2-(1.6647e-001)\,u^3 -(5.8641e-004)\,u^4+(9.2222e-003)\,u^5-(6.6455e-004)\,u^6$$

and

$$y_8(u)=(1.0000e+000)\,u+(1.5357e-009)\,u^2-(1.6667e-001)\,u^3 +(1.3459e-007)\,u^4+(8.3329e-003)\,u^5+(1.0176e-006)\,u^6 -(1.9974e-003)\,u^7+(9.9488e-007)\,u^8+(2.4041e-006)\,u^9.$$

The numerical values of $y(u) = \sin(u)$ and $y_M(u)$ can be seen in Table 1. Also, the numerical values of the error fuctions $e_M(u)$ for Bessel polynomial approach method (BPAM) of [53] and for the present method can be seen in Table 2. Additionally, for a visual comparison of the error functions, you can refer to Figures 1–3.

Table 1. Numerical values of $y(u)$ and $y_M(u)$ of equation (25) for the present method.

u_i	$y(u_i) = sin(u_i)$	$y_3(u_i)$	$y_5(u_i)$	$y_8(u_i)$
0	0	0	0	0
0.2	0.198669330795	0.198753775035	0.198669137704	0.198669330801
0.4	0.389418342309	0.389544075711	0.389418171174	0.389418342327
0.6	0.564642473395	0.564742579305	0.564642223290	0.564642473450
0.8	0.717356090900	0.717491914996	0.717355816631	0.717356091050
1	0.841470984808	0.841705663862	0.841470450399	0.841470985182

Table 2. Numerical values of $e_M(u_i)$ of equation (25) for the methods.

u_i	BPAM for $N=3$	BPAM for $N=7$	Present method for $N=3$	Present method for $N=7$
0	$1.2530e-17$	0	0	0
0.2	$1.1506e-004$	$7.8343e-010$	$8.4444e-005$	$2.2596e-010$
0.4	$1.7847e-004$	$9.2046e-010$	$1.2573e-004$	$2.6435e-010$
0.6	$5.1456e-005$	$1.0483e-009$	$1.0011e-004$	$3.3941e-010$
0.8	$2.1418e-004$	$9.7238e-009$	$1.3582e-004$	$4.6650e-010$
1	$2.0442e-003$	$5.4000e-007$	$2.3468e-004$	$8.0256e-010$

Example 5.2 [14] *Let us see the following problem*

$$y'(u) = \frac{1}{2}y(u) + \frac{1}{2}e^{\frac{u}{2}}y\left(\frac{u}{2}\right),\ 0 \le u \le 1 \tag{26}$$

and the initial condition

$$y(0) = 1.$$

The exact solution of this problem is $y(u) = e^u$.
Comparing the equations of this example by equations (1) and (2) it can be easily seen that $H(u) = \frac{1}{2}$, $P_1(u) = \frac{1}{2}e^{\frac{u}{2}}$, $\lambda_1 = \frac{1}{2}$, $a = 0$, $\gamma = 1$ and $g(u) = 0$. Also, the following basic matrix equation can be obtained by using equation (16) as

$$\left\{\mathbf{U}\mathbf{D}^T - \mathbf{H}\bar{\mathbf{U}}\mathbf{U}\mathbf{M}\mathbf{D}^T - \lambda_1\mathbf{P}_1\bar{\mathbf{U}}\mathbf{U}\mathbf{B}(\lambda_1)\mathbf{M}\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G}.$$

Using the present method for $M=3$, $M=6$, and $M=8$, the approximate solutions can be obtained as

$$y_3(u) = 1 + (1.0000e+00)\,u + (5.0743e-01)\,u^2 + (1.4141e-01)\,u^3 + (6.9888e-02)\,u^4,$$

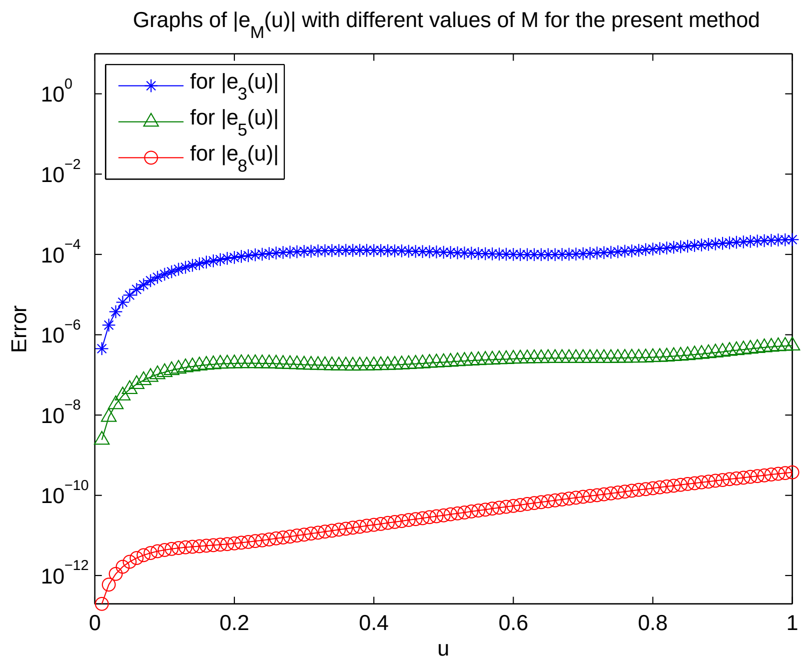

Figure 1. Comparison of the error function $e_M\left(u\right)$ for equation (25) when $M = 3, 5, 8$.

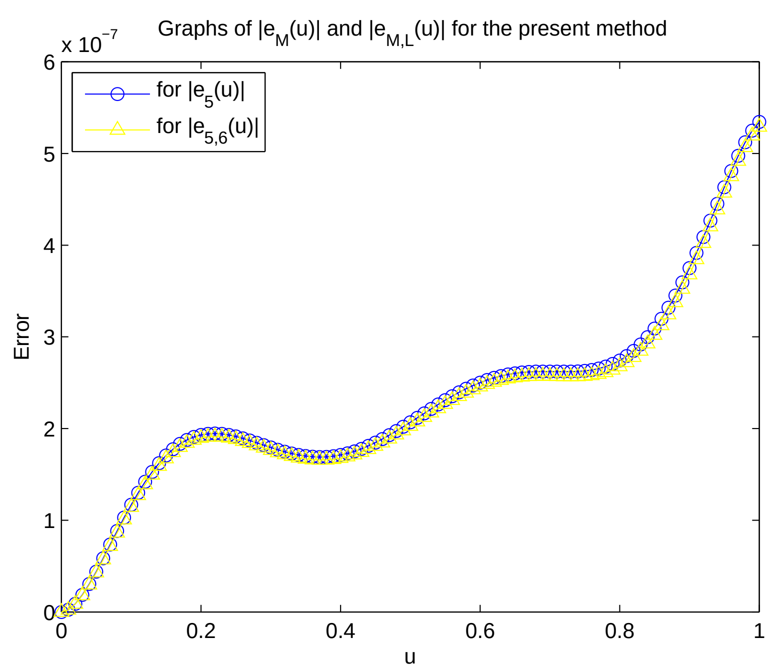

Figure 2. Comparison of the error functions $e_M(u)$ and $E_{M,L}(u)$ for equation (25) when $M = 5$ and $L = 6$.

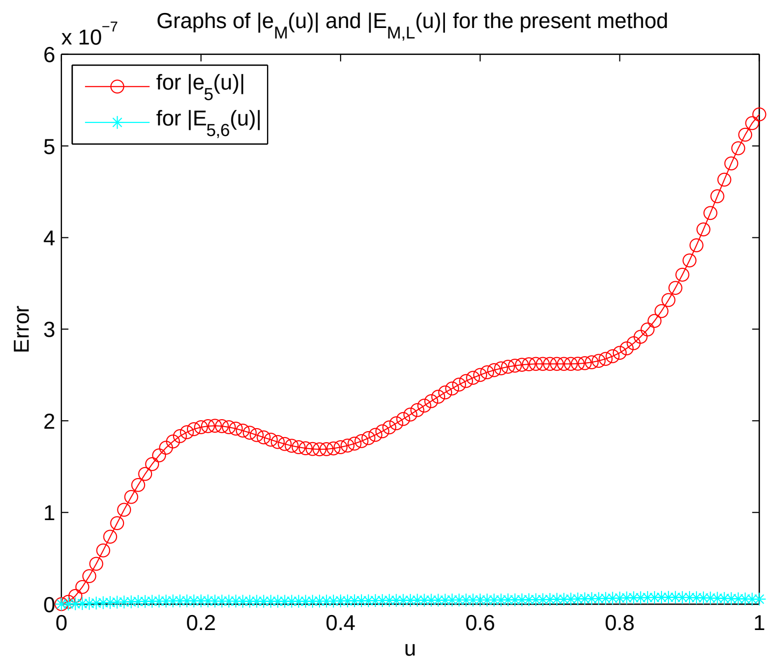

Figure 3. Comparison of error functions $e_M(u)$ and $e_{M,L}(u)$ for equation (25) when $M = 5$ and $L = 6$.

$$y_6(u) = 1 + (1.0000e + 000)\,u + (5.0000e - 001)\,u^2 + (1.6669e - 001)\,u^3 + (4.1571e - 002)\,u^4 + (8.5363e - 003)\,u^5 + (1.1572e - 003)\,u^6 + (3.2973e - 004)\,u^7$$

and

$$y_8(u) = 1 + (1.0000e + 000)\,u + (5.0000e - 001)\,u^2 + (1.6667e - 001)\,u^3 + (4.1666e - 002)\,u^4 + (8.3351e - 003)\,u^5 + (1.3850e - 003)\,u^6 + (2.0362e - 004)\,u^7 + (2.0593e - 005)\,u^8 + (4.5700e - 006)\,u^9.$$

The numerical values of the error fuctions $e_M(u)$ for Spline method (SM) of [13], Taylor series method (TSM) of [38], and the present method can be compared by Table 3. Beside, comparison of the graphs of the solutions and the error functions can be seen in Figures 4–7, respectively.

Table 3. Numerical values of $e_M(u_i)$ of equation (26) for different methods.

u_i	SM for $m = 2$	TSM for $N = 8$	Present method for $M = 8$
0.2	$0.198e - 007$	$1.440e - 012$	$7.3663e - 012$
0.4	$0.473e - 007$	$1.440e - 012$	$6.2642e - 012$
0.6	$0.847e - 007$	$2.953e - 008$	$5.4516e - 011$
0.8	$0.135e - 006$	$4.018e - 007$	$1.8917e - 010$
1	$0.201e - 006$	$3.059e - 006$	$5.2791e - 010$

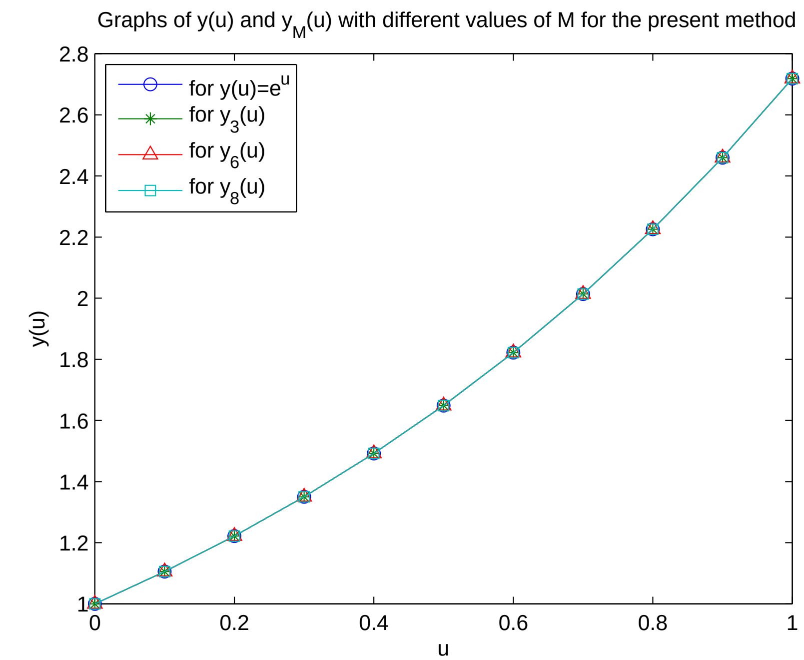

Figure 4. Comparison of the solutions for equation (26) when $M = 3, 6, 8$.

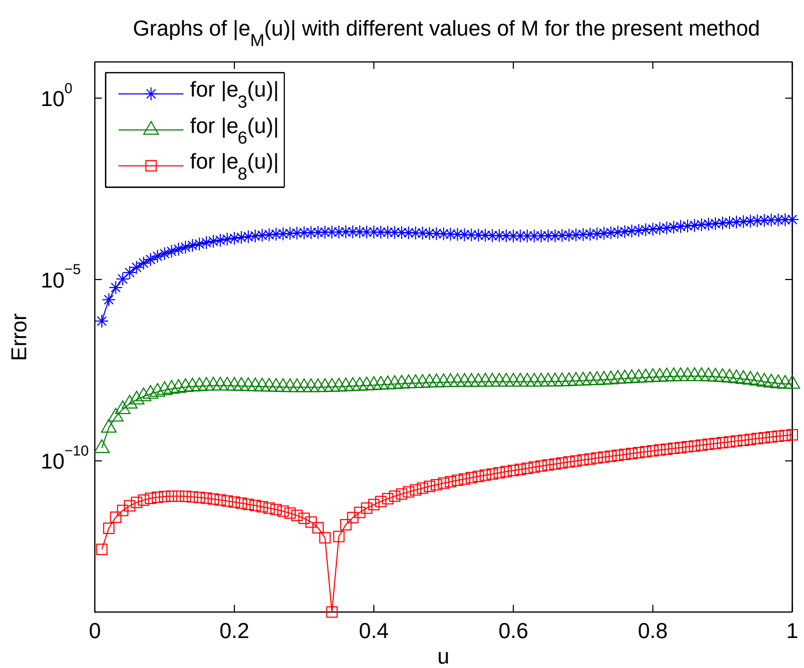

Figure 5. Comparison of the error function $e_M(u)$ for equation (26) when $M = 3, 6, 8$.

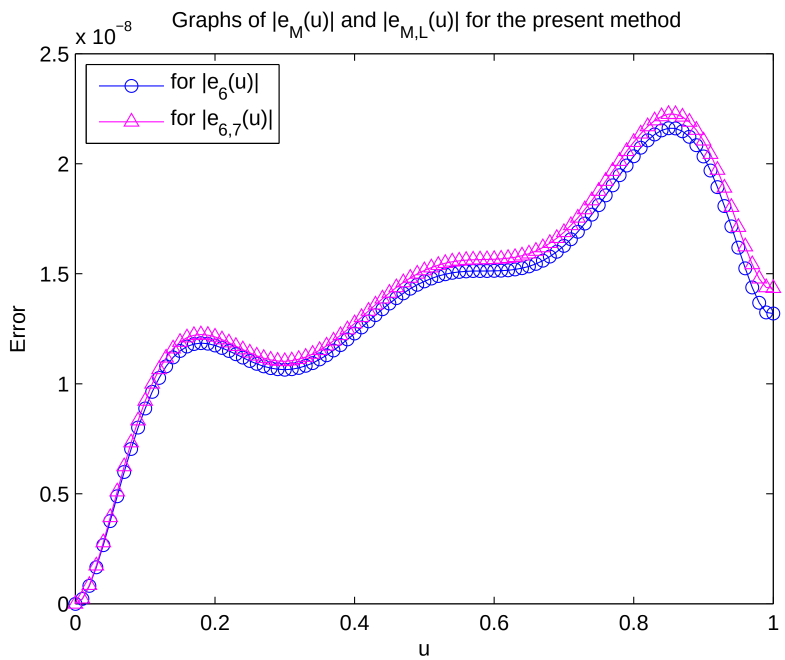

Figure 6. Comparison of error functions $e_M(u)$ and $e_{M,L}(u)$ for equation (26) when $M=6$ and $L=7$.

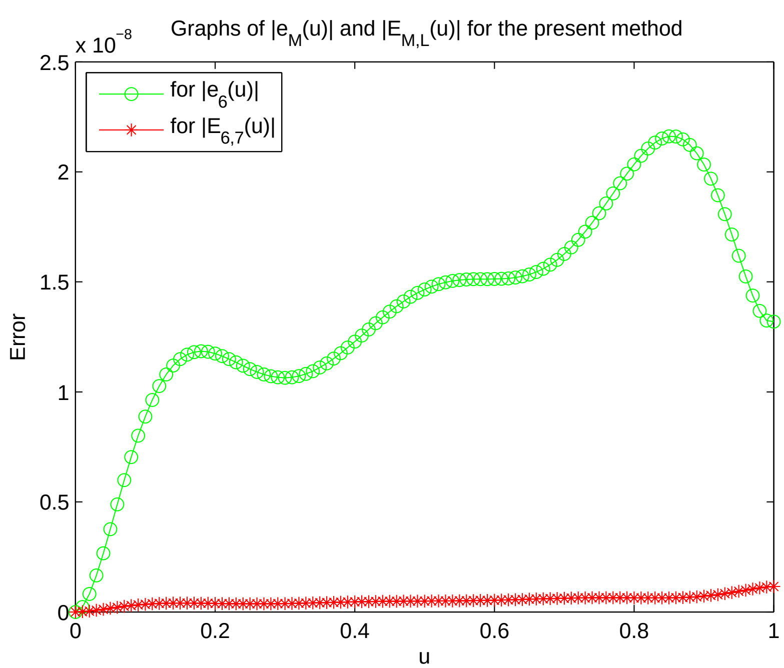

Figure 7. Comparison of error functions $e_M(u)$ and $E_{M,L}(u)$ for equation (26) when $M=6$ and $L=7$.

Example 5.3 [7] *Now, let us look for the following problem*

$$y'(u) = -y(u) + 0.1y(0.8u) + 0.5y'(0.8u) + (0.32u - 0.5)e^{(-0.8u)} + e^{(-u)},\ 0 \le u \le 1 \tag{27}$$

and the initial condition

$$y(0) = 0.$$

The exact solution is $y(u) = ue^{-u}$ *for this problem. If one compares the equations of this example by equations (1) and (2), then it can be easily seen that* $H(u) = -1$, $P_1(u) = 0.1$, $\lambda_1 = 0.8$, $Q_1(u) = 0.5$, $\mu_1 = 0.8$, $g(u) = (0.32u - 0.5)e^{(-0.8u)} + e^{(-u)}$, $a = 0$, *and* $\gamma = 0$. *Also, we have the following fundamental matrix equation by using equation (16) as*

$$\left\{\mathbf{U}\mathbf{D}^T - \mathbf{H}\bar{\mathbf{U}}\mathbf{U}\mathbf{M}\mathbf{D}^T - \lambda_1\mathbf{P}_1\bar{\mathbf{U}}\mathbf{U}\mathbf{B}(\lambda_1)\mathbf{M}\mathbf{D}^T - \mathbf{Q}_1\mathbf{U}\mathbf{B}(\mu_1)\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G}.$$

The numerical values of the error function $e_M(u)$ *where* $M = 4$ *and* $M = 6$, $e_{M,L}(u)$ *and* $E_{M,L}(u)$ *for* $M = 6$ *and* $L = 7$ *can be seen in Table 4. Also, the graphs of the solutions can be seen in Figure 8. Beside, the graphs of the error functions* $e_M(u)$ *of different methods which are TSORKM for two-stage order-one Runge-Kutta method of [3], BWM for Bernoulli wavelets method of [25], BPAM for Bessel polynomial approach method of [53], and SM for spectral method of [36] can be compared with the present method in Figure 9.*

Table 4. Numerical values of $e_M(u_i)$, $e_{M,L}(u_i)$ and $E_{M,L}(u_i)$ of equation (27) for the present method.

u_i	$e_4(u_i)$	$e_6(u_i)$	$e_{6,7}(u_i)$	$E_{6,7}(u_i)$
0	0	0	0	0
0.2	$2.0798e-005$	$5.3020e-008$	$-5.0939e-008$	$2.0814e-009$
0.4	$1.6677e-005$	$3.2323e-008$	$-3.0983e-008$	$1.3410e-009$
0.6	$8.8444e-006$	$2.1387e-008$	$-2.0540e-008$	$8.4676e-010$
0.8	$8.0680e-006$	$1.6528e-008$	$-1.6055e-008$	$4.7338e-010$
1	$4.2855e-006$	$1.1660e-008$	$1.2802e-008$	$1.1436e-009$

Example 5.4 [51] *Let us consider the following pantograph equation*

$$y'(u) = -y(u) + 0.5y(0.5u) + 0.5y'(0.5u),\ 0 \le u \le 1 \tag{28}$$

and the initial condition

$$y(0) = 1.$$

The exact solution of this problem is $y(u) = e^{-u}$. *Comparing this initial value problem by the (1)-(2) initial value problem, we can see* $H(u) = -1$, $P_1(u) = 0.5$, $\lambda_1 = 0.5$, $Q_1(u) = 0.5$, $\mu_1 = 0.5$, $g(u) = 0$, $a = 0$, *and* $\gamma = 1$. *Moreover, the fundamental matrix equation of this problem can be obtained by equation (16) as*

$$\left\{\mathbf{U}\mathbf{D}^T - \mathbf{H}\bar{\mathbf{U}}\mathbf{U}\mathbf{M}\mathbf{D}^T - \lambda_1\mathbf{P}_1\bar{\mathbf{U}}\mathbf{U}\mathbf{B}(\lambda_1)\mathbf{M}\mathbf{D}^T - \mathbf{Q}_1\mathbf{U}\mathbf{B}(\mu_1)\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G} + \bar{\mathbf{H}} + \bar{\mathbf{P}}_i.$$

The numerical values of different methods which are TSORKM for two-stage order-one Runge-Kutta method of [3], OLM for one-leg θ*-method of [46, 47], VIM for variational iteration method of [7], and BWM for Bernoulli*

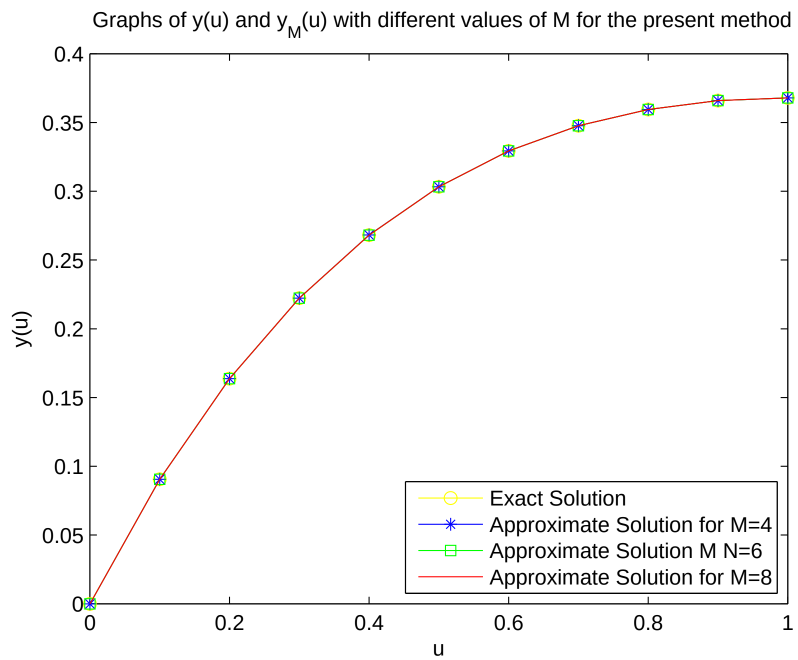

Figure 8. Comparison of the solutions for equation (27) when $M = 4, 6, 8$.

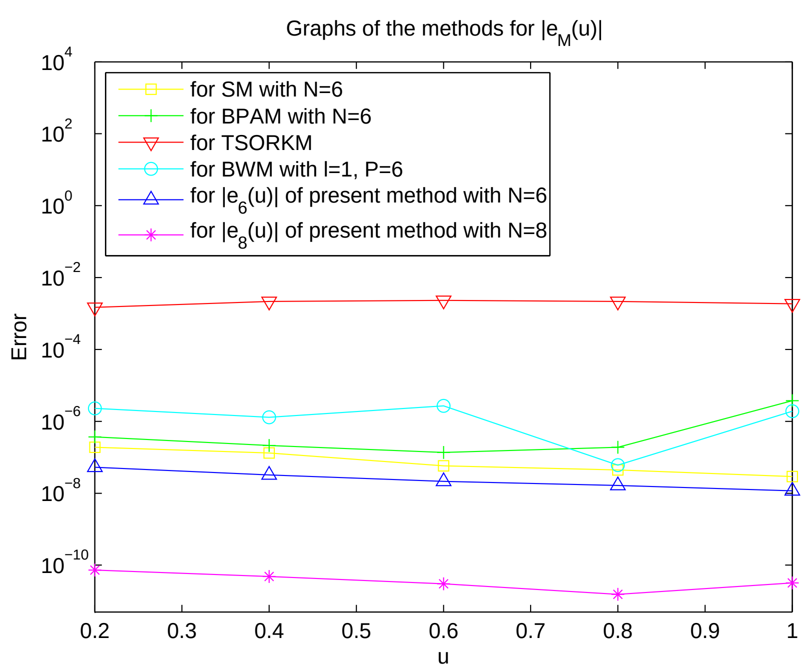

Figure 9. Comparison of $e_M(u)$ of the methods for equation (27).

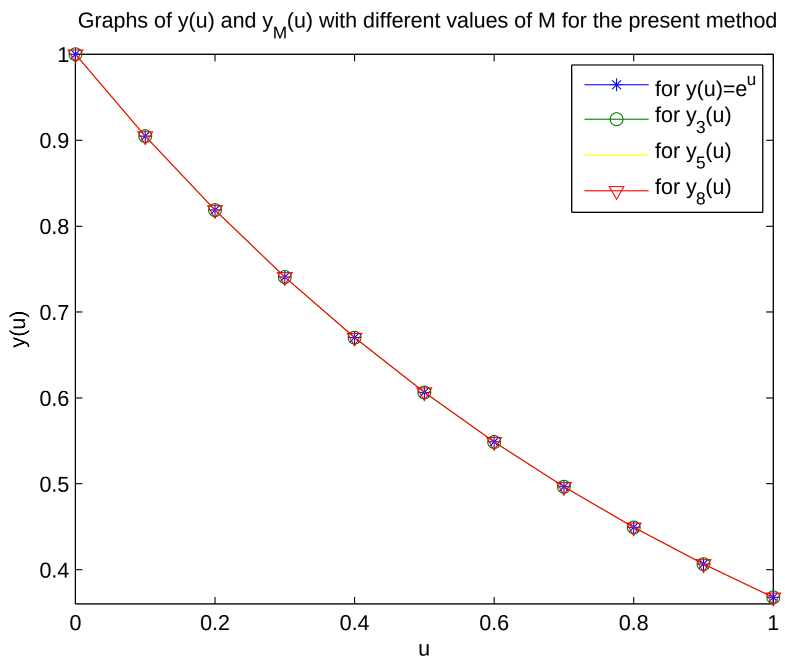

Figure 10. Comparison of the solutions for equation (28) when $M = 3, 5, 8$.

wavelets method of [25] can be compared with the present method in Table (5).Also, the graphs of the solutions can be seen in Figure 10. Additionally, the comparison of the graphs of $e_M(u)$, $e_{M,L}(u)$, and $E_{M,L}(u)$ for the present method can be seen in Figures 11 and 12.

Table 5. Numerical values of $e_M(u_i)$ of the present method and the other methods for equation (28).

u_i	TSORKM	OLM for $\theta = 0.8$	VIM for $N = 7$	BWM for l=1, P=6	Present method for $M = 5$
0.2	$8.24e-04$	$8.86e-03$	$7.08e-04$	$2.37e-06$	$2.0022e-007$
0.4	$1.35e-03$	$2.66e-02$	$1.29e-03$	$2.46e-06$	$2.3182e-007$
0.6	$1.66e-03$	$4.58e-02$	$1.76e-03$	$2.10e-06$	$2.5244e-007$
0.8	$1.81e-03$	$6.29e-02$	$2.15e-03$	$1.73e-06$	$2.0283e-007$
1	$1.85e-03$	$7.66e-02$	$2.47e-03$	$1.48e-06$	$3.0437e-007$

Example 5.5 [44] *Now, let us consider the following first order linear pantograph equation*

$$y'(u) = -y(0.8u) - y(u), \; 0 \leq u \leq 1 \tag{29}$$

with the initial condition

$$y(0) = 1.$$

The exact solution of this problem does not exist. Therefore, the approximate solutions of different methods can be compared with the present method in Table 6. Now, if we compare the equations of this example by the (1)

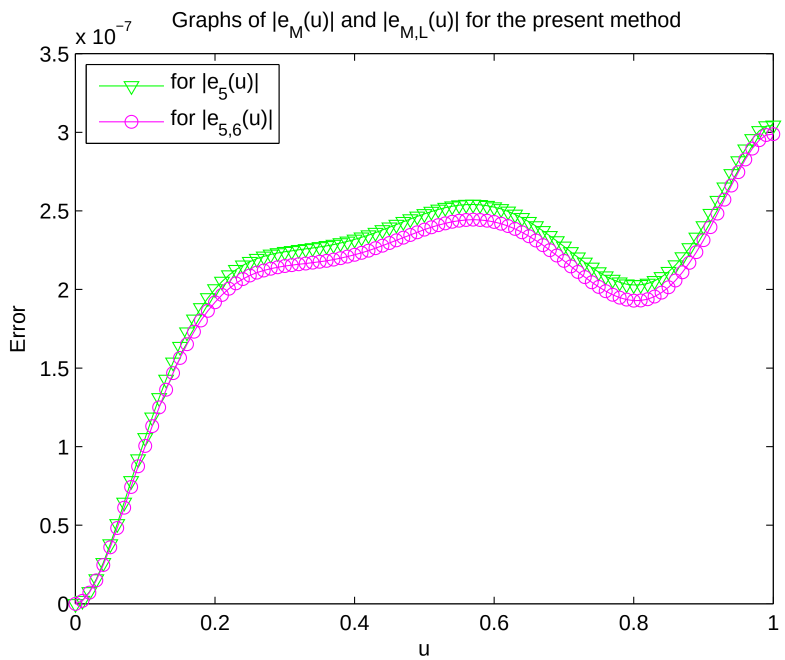

Figure 11. Comparison of error functions $e_M(u)$ and $e_{M,L}(u)$ for equation (28) when $M=5$ and $L=6$.

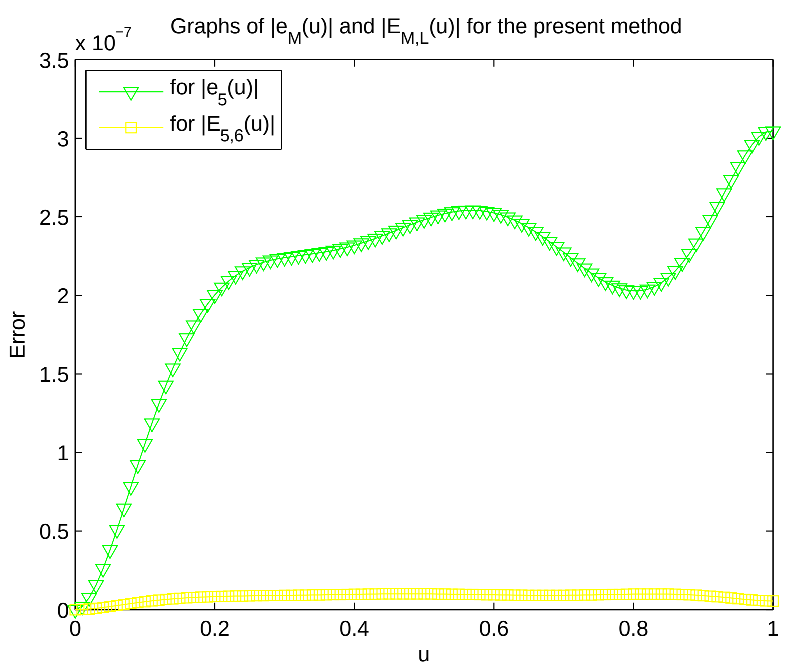

Figure 12. Comparison of error functions $e_M(u)$ and $E_{M,L}(u)$ for equation (28) when $M=5$ and $L=6$.

- (2) {initial value problem} then it can be easily seen that $H(u) = -1$, $P_1(u) = -1$, $\lambda_1 = 0.8$, $g(u) = 0$, $a = 0$, and $\gamma = 1$. Then the fundamental matrix equation can be obtained by equation (16) as

$$\left\{\mathbf{U}\mathbf{D}^T - \mathbf{H}\bar{\mathbf{U}}\mathbf{U}\mathbf{M}\mathbf{D}^T - \lambda_1\mathbf{P}_1\bar{\mathbf{U}}\mathbf{U}\mathbf{B}(\lambda_1)\mathbf{M}\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G} + \bar{\mathbf{H}} + \bar{\mathbf{P}}_i.$$

The numerical values of different methods which are WSM for Walsh series method of [34], LSM for Laguerre series method of [23], TSM for Taylor series method of [40], HSM for Hermit series method of [49], and CMBOM for collocation method based on Bernoulli operational matrix of [44] can be compared with the approximation of the present method in Table 6. Moreover, the graphs of the approximate solutions of the present method, as well as the graphs of the error functions $e_{M,L}(u)$ with distinct values of M and L, might be compared by referring to Figures 13 and 14, respectively.The graphs in Figure 14 are generated using the method outlined in Section 4, illustrating how accurately the present method approximates the solution to the problem.

Table 6. Numerical values of the approximations of different methods for the equation (29).

u_i	WSM	LSM for $N = 20$	TSM for $N = 8$	HSM for $N = 8$	CMBOM for $N = 6$	Present method for $M = 6$
0	1	0.999971	1	1	1	1
0.2	0.665621	0.664703	0.6664691	0.664691	0.66469052	0.664690929
0.4	0.432426	0.433555	0.433561	0.433561	0.43356055	0.433560744
0.6	0.275140	0.276471	0.276482	0.276482	0.27648223	0.276482309
0.8	0.170320	0.171482	0.171484	0.171484	0.17148362	0.171484083
1	0.100856	0.102679	0.102744	0.102670	0.10268323	0.102670192

Example 5.6 *Finally, let's consider the following problem*

$$y'(u) = y(u) - 0.5y(0.2u) + 2u - 0.98u^2 - 1,\ 0 \leq u \leq 1 \tag{30}$$

with the initial condition

$$y(0) = 1.$$

This problem has the polynomial exact solution $y(u) = u^2 + 2$. *Comparing this initial value problem by the*

(1)-(2) initial value problem, we can see $H(u) = 1$, $P_1(u) = -0.5$, $\lambda_1 = 0.2$, $g(u) = 2u - 0.98u^2 - 1$, $a = 0$, *and* $\gamma = 1$. *By this information, the following fundamental matrix equation can be obtained by using equation (16) as*

$$\left\{\mathbf{U}\mathbf{D}^T - \mathbf{H}\bar{\mathbf{U}}\mathbf{U}\mathbf{M}\mathbf{D}^T - \lambda_1\mathbf{P}_1\bar{\mathbf{U}}\mathbf{U}\mathbf{B}(\lambda_1)\mathbf{M}\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G} + \bar{\mathbf{H}} + \bar{\mathbf{P}}_i.$$

For this problem, the present method gives the exact solution when $M = 3$. *Thus, this problem shows that the exact solution of a problem whose exact solution is polynomial can be obtained by applying the method of this work.*

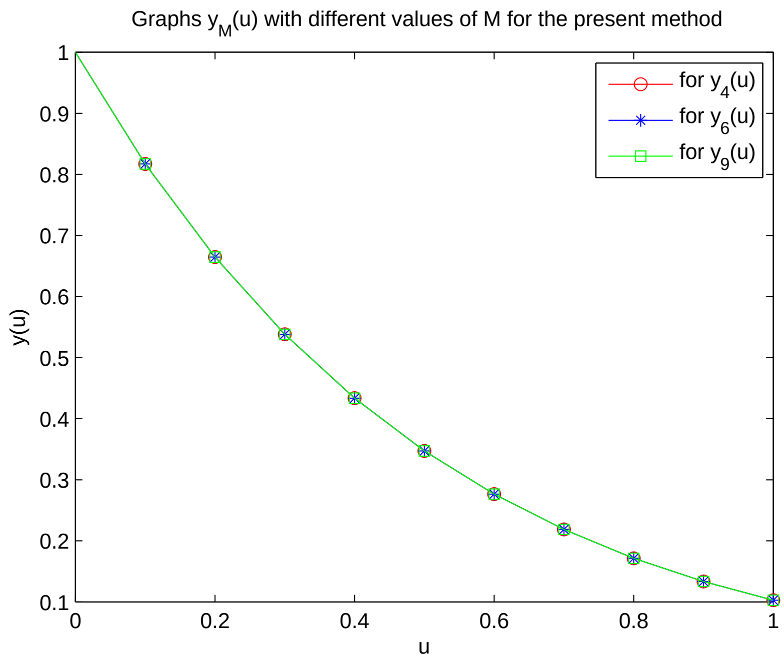

Figure 13. Comparison of the aproximate solutions of the present method for equation (29) when $M = 4, 6, 9$.

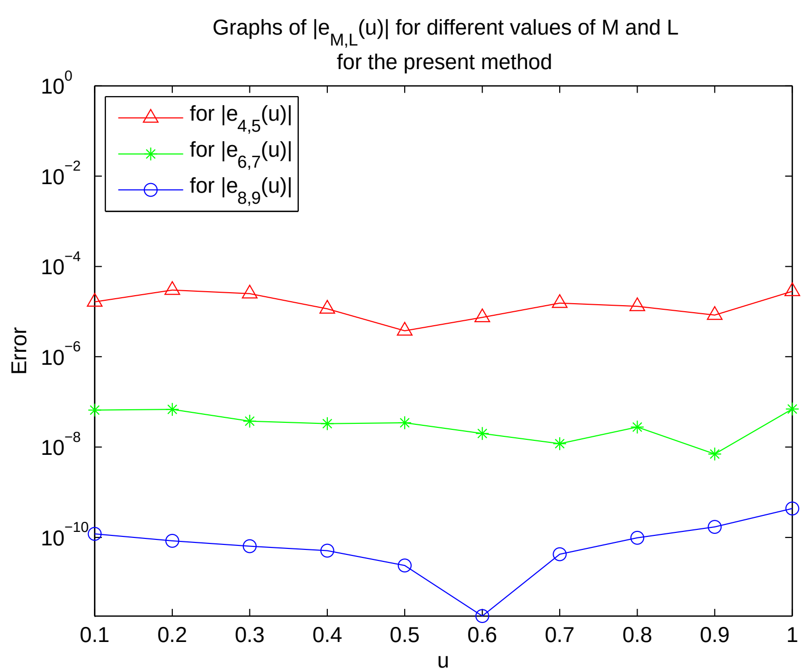

Figure 14. Comparison of the methods for the error function $e_{M,L}(u)$ of equation (29) for different values of M and L.

6. Conclusion

Finding solutions for most linear neutral delay differential equations is not always straightforward. As a result, researchers have turned to polynomial approximation methods in the literature to obtain approximate solutions for these problems. Many of these methods, which employ the collocation method, rely on differential relations. However, in this study, a novel approximate method is presented that utilizes the series expansion of Clique polynomials and incorporates integral relations. This leads to a polynomial approximation with a higher degree compared to other methods for the same value of M. This is because the degree of the polynomials increases when integration is employed, resulting in more accurate results than other approximation methods found in the literature. This can be observed in the comparisons made in Section 5 with the results obtained from other methods. Furthermore, the error estimation technique outlined in Section 4 is crucial as it yields results that closely align with the actual errors. This estimation serves as a valuable metric for assessing the reliability of results, particularly for problems with unknown exact solutions. Additionally, this estimated error function can be used to refine solutions and subsequently reduce errors. In conclusion, this study significantly advances our understanding of neutral delay differential equations, providing effective and efficient results.

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