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Numerical solutions of the Kawahara equation by the septic B-spline collocation method

Battal Gazi Karakoç[1, *], Halil Zeybek[2] and Turgut Ak[3]

[1] *Nevsehir Haci Bektas Veli University, Nevsehir, 50300, Turkey*
[2] *Abdullah Gul University, Kayseri, 38039, Turkey*
[3] *Armutlu Vocational High School, Yalova University, 77500, Yalova, Turkey*

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Abstract In this article, a numerical solution of the Kawahara equation is presented by septic B-spline collocation method. Applying the Von-Neumann stability analysis, the present method is shown to be unconditionally stable. The accuracy of the proposed method is checked by two test problems. L_2 and L_∞ error norms and conserved quantities are given at selected times. The obtained results are found in good agreement with the some recent results.

Keywords Kawahara equation, Finite element method, Septic B-Splines, Collocation, Solitary waves.

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1. Introduction

In this paper, we consider the Kawahara equation which is firstly studied by Kawahara[1],

$$U_t + (\frac{105}{16})\alpha^2 U U_x + (\frac{13}{4})\beta U_{xxx} - U_{xxxxx} = 0, \qquad (1)$$

*Correspondence to: Nevsehir Haci Bektas Veli University, Nevsehir, 50300, Turkey. E-mail: sbgk44@gmail.com.

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where α, β are nonzero arbitrary constants. This equation is also known as a fifth-order KdV equation or singularly perturbed KdV equation[2]. The fifth-order KdV equation is one of the most known nonlinear evolution equation which is used in the theory of magneto-acoustic waves in a plasma, capillary-gravity waves and in the theory of shallow water waves having surface tension. If the coefficient of the term having third-order derivative is dominant over that of the fifth-order, a monotone solitary wave solution is found. But if the fifth-order derivative is dominating over the third one, oscillatory structure of the solitary waves forms which are called as Kawahara solitons[3]. If we take $\alpha = 4/\sqrt{105}$ and $\beta = 4/13$, Eq.(1) returns to another form of the equation

$$U_t + UU_x + U_{xxx} - U_{xxxxx} = 0. \tag{2}$$

Various analytical and numerical studies including; Crank-Nicolson Differential quadrature method[3], RBF collocation method[4], meshless method of lines[5], Dual-Petrov Galerkin method[6], Adomian decomposition method[7], tanh-function method[8], variational iteration and homotopy perturbation method[9] and sine-cosine method[10] have been proposed for solving the Kawahara type equations.

In the present paper, a numerical scheme based on the septic B- spline collocation method has been set up for solving the Kawahara equation with a variant of initial and boundary conditions. This paper is organized as follows: In Section 2, numerical algorithm is presented. In Section 3, stability analysis of the scheme is given and numerical results of the equation are obtained in Section 4. Finally a summary is presented in Section 5.

2. Septic B-spline finite element solution

Consider the Kawahara Eq.(2) with the following boundary and initial conditions

$$U(a,t) = \alpha_1, \quad U(b,t) = \alpha_2, \quad a \le x \le b, \quad t > 0, \tag{3}$$

$$U(x,0) = f(x). \tag{4}$$

To implement the numerical algorithm, solution region of the problem is restricted over an interval $a \le x \le b$. And let the partition of the space interval $[a,b]$ into uniformly sized finite elements of length h by the knots x_m such that $a = x_0 < x_1 < ... < x_N = b$ and $h = \frac{b-a}{N}$. The set of septic B-spline functions $\{\phi_{-3}(x), \phi_{-2}(x), \dots, \phi_{N+2}(x), \phi_{N+3}(x)\}$ forms a basis over the problem domain of solution $[a,b]$. A global approximation $U_N(x,t)$ is expressed in terms of the septic B-splines as

$$U_N(x,t) = \sum_{i=-3}^{N+3} \phi_i(x)\delta_i(t) \tag{5}$$

where $\delta_i(t)$ are time dependent parameters to be determined from the boundary and collocation conditions. Septic B-splines $\phi_m(x)$, (m= $-3, -2, ..., N+2, N+3$) at the knots x_m are defined over the interval $[a, b]$ by[11]

$$\phi_m(x) = \frac{1}{h^7}\begin{cases} (x-x_{m-4})^7 & [x_{m-4}, x_{m-3}] \\ (x-x_{m-4})^7 - 8(x-x_{m-3})^7 & [x_{m-3}, x_{m-2}] \\ (x-x_{m-4})^7 - 8(x-x_{m-3})^7 + 28(x-x_{m-2})^7 & [x_{m-2}, x_{m-1}] \\ (x-x_{m-4})^7 - 8(x-x_{m-3})^7 + 28(x-x_{m-2})^7 - 56(x-x_{m-1})^7 & [x_{m-1}, x_m] \\ (x_{m+4}-x)^7 - 8(x_{m+3}-x)^7 + 28(x_{m+2}-x)^7 - 56(x_{m+1}-x)^7 & [x_m, x_{m+1}] \\ (x_{m+4}-x)^7 - 8(x_{m+3}-x)^7 + 28(x_{m+2}-x)^7 & [x_{m+1}, x_{m+2}] \\ (x_{m+4}-x)^7 - 8(x_{m+3}-x)^7 & [x_{m+2}, x_{m+3}] \\ (x_{m+4}-x)^7 & [x_{m+3}, x_{m+4}] \\ 0 & otherwise. \end{cases} \tag{6}$$

Substituting trial function (6) into Eq.(5), the nodal values of $U, U', U'', U''', U^{iv}, U^v$ at the knots x_m are obtained in terms of the element parameters δ_m by

$$\begin{aligned} U_N(x_m,t) = U_m &= \delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + \\ &\quad 120\delta_{m+2} + \delta_{m+3}, \quad m = 0, ..., N, \\ U'_m &= \tfrac{7}{h}(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}), \\ U''_m &= \tfrac{42}{h^2}(\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_m + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3}), \\ U'''_m &= \tfrac{210}{h^3}(-\delta_{m-3} - 8\delta_{m-2} + 19\delta_{m-1} - 19\delta_{m+1} + 8\delta_{m+2} + \delta_{m+3}), \\ U^{iv}_m &= \tfrac{840}{h^4}(\delta_{m-3} - 9\delta_{m-1} + 16\delta_m - 9\delta_{m+1} + \delta_{m+3}), \\ U^v_m &= \tfrac{2520}{h^5}(-\delta_{m-3} + 4\delta_{m-2} - 5\delta_{m-1} + 5\delta_{m+1} - 4\delta_{m+2} + \delta_{m+3}). \end{aligned} \tag{7}$$

The B-splines $\phi_m(x)$ and its six principle derivatives vanish outside the interval $[x_{m-4}, x_{m+4}]$.

Now we identify the collocation points with the knots and use Eq.(7) to evaluate U_m, its necessary space derivatives and substitute into Eq.(2) to obtain the set of the coupled ordinary differential equations:

$$\begin{aligned} &\dot{\delta}_{m-3} + 120\dot{\delta}_{m-2} + 1191\dot{\delta}_{m-1} + 2416\dot{\delta}_m + 1191\dot{\delta}_{m+1} + 120\dot{\delta}_{m+2} + \\ &\tfrac{7Z_m}{h}(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}) + \\ &\tfrac{210}{h^3}(-\delta_{m-3} - 8\delta_{m-2} + 19\delta_{m-1} - 19\delta_{m+1} + 8\delta_{m+2} + \delta_{m+3}) - \\ &\tfrac{2520}{h^5}(-\delta_{m-3} + 4\delta_{m-2} - 5\delta_{m-1} + 5\delta_{m+1} - 4\delta_{m+2} + \delta_{m+3}) = 0, \end{aligned} \tag{8}$$

where

$$Z_m = U_m = \\ \delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3},$$

and $\cdot$ denotes derivative with respect to time. If time parameters δ_i's and its time derivatives $\dot{\delta}_i$'s in Eq.(8) are discretized by the Crank-Nicolson formula and usual finite difference aproximation, respectively,

$$\delta_i = \frac{1}{2}(\delta^n + \delta^{n+1}), \qquad \dot{\delta}_i = \frac{\delta^{n+1} - \delta^n}{\Delta t} \tag{9}$$

we obtain a recurrence relationship between two time levels n and $n+1$ relating two unknown parameters δ_i^{n+1}, δ_i^n for $i = m-2, m-1, ..., m+1, m+2$

$$\begin{aligned} & \gamma_1\delta_{m-3}^{n+1} + \gamma_2\delta_{m-2}^{n+1} + \gamma_3\delta_{m-1}^{n+1} + \gamma_4\delta_m^{n+1} + \gamma_5\delta_{m+1}^{n+1} + \gamma_6\delta_{m+2}^{n+1} + \gamma_7\delta_{m+3}^{n+1} \\ = \; & \gamma_8\delta_{m-3}^{n+1} + \gamma_9\delta_{m-2}^{n+1} + \gamma_{10}\delta_{m-1}^{n+1} + \gamma_{11}\delta_m^{n+1} + \gamma_{12}\delta_{m+1}^{n+1} + \gamma_{13}\delta_{m+2}^{n+1} + \gamma_{14}\delta_{m+3}^{n+1}, \end{aligned} \tag{10}$$

where

$$\begin{array}{ll} \gamma_1 = (1 - EZ_m - M + K), & \gamma_2 = (120 - 56EZ_m - 8M - 4K), \\ \gamma_3 = (1191 - 245EZ_m + 19M + 5K), & \gamma_4 = (2416), \\ \gamma_5 = (1191 + 245EZ_m - 19M - 5K), & \gamma_6 = (120 + 56EZ_m + 8M + 4K), \\ \gamma_7 = (1 + EZ_m + M - K), & \gamma_8 = (1 + EZ_m + M - K), \\ \gamma_9 = (120 + 56EZ_m + 8M + 4K), & \gamma_{10} = (1191 + 245EZ_m - 19M - 5K), \\ \gamma_{11} = (2416), & \gamma_{12} = (1191 - 245EZ_m + 19M + 5K), \\ \gamma_{13} = (120 - 56EZ_m - 8M - 4K), & \gamma_{14} = (1 - EZ_m - M + K) \\ \multicolumn{2}{c}{m = 0, 1, \ldots, N, \quad E = \frac{7}{2h}\Delta t, \quad M = \frac{105}{h^3}\Delta t, \quad K = \frac{1260}{h^5}\Delta t.} \end{array} \tag{11}$$

The system (10) consists of $(N+1)$ linear equations including $(N+7)$ unknown parameters $(\delta_{-3}, \delta_{-2}, \delta_{-1} \ldots, \delta_{N+1}, \delta_{N+2}, \delta_{N+3})^T$. To obtain a unique solution to this system, we need six additional constraints. These are obtained from the boundary conditions and can be used to eliminate $\delta_{-3}, \delta_{-2}, \delta_{-1}$ and $\delta_{N+1}, \delta_{N+2}, \delta_{N+3}$ from the system (10) which then becomes a matrix equation for the $N+1$ unknowns $d^n = (\delta_0, \delta_1, \ldots, \delta_N)^T$ of the form

$$A\mathbf{d^{n+1}} = B\mathbf{d^n}. \tag{12}$$

The matrices A and B are $(N+1) \times (N+1)$ septa-diagonal matrices given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & & & & & & \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & & & & & \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & & & & \\ & & & & \ddots & & & & & \\ & & a_{m,m-3} & a_{m,m-2} & a_{m,m-1} & a_{m,m} & a_{m,m+1} & a_{m,m+2} & a_{m,m+3} & \\ & & & & & \ddots & & & & \\ & & & & a_{n-1,n-4} & a_{n-1,n-3} & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} \\ & & & & & a_{n,n-3} & a_{n,n-2} & a_{n,n-1} & a_{n,n} & a_{n,n+1} \\ & & & & & & a_{n+1,n-2} & a_{n+1,n-1} & a_{n+1,n} & a_{n+1,n+1} \end{bmatrix},$$

and, for $m = 4, 5, ...n-2$,

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & & & & & & \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & & & & & \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & & & & \\ & & & & \ddots & & & & & \\ & & b_{m,m-3} & b_{m,m-2} & b_{m,m-1} & b_{m,m} & b_{m,m+1} & b_{m,m+2} & b_{m,m+3} & \\ & & & & & \ddots & & & & \\ & & & & b_{n-1,n-4} & b_{n-1,n-3} & b_{n-1,n-2} & b_{n-1,n-1} & b_{n-1,n} & b_{n-1,n+1} \\ & & & & & b_{n,n-3} & b_{n,n-2} & b_{n,n-1} & b_{n,n} & b_{n,n+1} \\ & & & & & & b_{n+1,n-2} & b_{n+1,n-1} & b_{n+1,n} & b_{n+1,n+1} \end{bmatrix}$$

$$\begin{aligned}
a_{11} &= \frac{186032}{297}EZ_m - \frac{108720}{297}K, & b_{11} &= -\frac{186032}{297}EZ_m + \frac{108720}{297}K,\\
a_{12} &= \frac{183414}{297}EZ_m - \frac{107190}{297}K, & b_{12} &= -\frac{183414}{297}EZ_m + \frac{107190}{297}K,\\
a_{13} &= \frac{18480}{297}EZ_m - \frac{10800}{297}K, & b_{13} &= -\frac{18480}{297}EZ_m + \frac{10800}{297}K,\\
a_{14} &= \frac{154}{297}EZ_m - \frac{90}{297}K, & b_{14} &= -\frac{154}{297}EZ_m + \frac{90}{297}K,\\
a_{21} &= \frac{215411}{297} - \frac{45585}{297}EZ_m + \frac{8663}{297}M + \frac{12961}{297}K, & b_{21} &= \frac{215411}{297} + \frac{45585}{297}EZ_m - \frac{8663}{297}M - \frac{12961}{297}K,\\
a_{22} &= \frac{616822.5}{297} + \frac{18481.5}{297}EZ_m + \frac{601.5}{297}M + \frac{10126.5}{297}K, & b_{22} &= \frac{616822.5}{297} - \frac{18481.5}{297}EZ_m - \frac{601.5}{297}M - \frac{10126.5}{297}K,\\
a_{23} &= \frac{340284}{297} + \frac{75168}{297}EZ_m - \frac{5640}{297}M - \frac{48}{297}K, & b_{23} &= \frac{340284}{297} - \frac{75168}{297}EZ_m + \frac{5640}{297}M + \frac{48}{297}K,\\
a_{24} &= \frac{35525.5}{297} + \frac{8338.5}{297}EZ_m + \frac{2378.5}{297}M - \frac{1197.5}{297}K, & b_{24} &= \frac{35525.5}{297} - \frac{8338.5}{297}EZ_m - \frac{2378.5}{297}M + \frac{1197.5}{297}K,\\
a_{25} &= 1 + EZ_m + M - K, & b_{25} &= 1 - EZ_m - M + K,\\
a_{31} &= \frac{34432}{297} - \frac{7108}{297}EZ_m - \frac{1168}{297}M - \frac{2396}{297}K, & b_{31} &= \frac{34432}{297} + \frac{7108}{297}EZ_m + \frac{1168}{297}M + \frac{2396}{297}K,\\
a_{32} &= \frac{352833}{297} - \frac{71871}{297}EZ_m + \frac{6537}{297}M + \frac{591}{297}K, & b_{32} &= \frac{352833}{297} + \frac{71871}{297}EZ_m - \frac{6537}{297}M - \frac{591}{297}K,\\
a_{33} &= \frac{717432}{297} + \frac{120}{297}EZ_m + \frac{120}{297}M - \frac{120}{297}K, & b_{33} &= \frac{717432}{297} - \frac{120}{297}EZ_m - \frac{120}{297}M + \frac{120}{297}K,\\
a_{34} &= \frac{353726}{297} + \frac{72766}{297}EZ_m - \frac{5642}{297}M - \frac{1486}{297}K, & b_{34} &= \frac{353726}{297} - \frac{72766}{297}EZ_m + \frac{5642}{297}M + \frac{1486}{297}K,\\
a_{35} &= 120 + 28EZ_m + 8M + 4K, & b_{35} &= 120 - 28EZ_m - 8M - 4K,\\
a_{36} &= 1 + EZ_m + M - K, & b_{36} &= 1 - EZ_m - M + K,
\end{aligned}$$

$$\begin{aligned}
a_{m,m-3} &= 1 - EZ_m - M + K, & b_{m,m-3} &= 1 + EZ_m + M - K,\\
a_{m,m-2} &= 120 - 28EZ_m - 8M - 4K, & b_{m,m-2} &= 120 + 28EZ_m + 8M + 4K,\\
a_{m,m-1} &= 1191 - 245EZ_m + 19M + 5K, & b_{m,m-1} &= 1191 + 245EZ_m - 19M - 5K,\\
a_{m,m} &= 2416, & b_{m,m} &= 2416,\\
a_{m,m+1} &= 1191 + 245EZ_m - 19M - 5K, & b_{m,m+1} &= 1191 - 245EZ_m + 19M + 5K,\\
a_{m,m+2} &= 120 + 28EZ_m + 8M + 4K, & b_{m,m+2} &= 120 - 28EZ_m - 8M - 4K,\\
a_{m,m+3} &= 1 + EZ_m + M - K, & b_{m,m+3} &= 1 - EZ_m - M + K,
\end{aligned}$$

Two or three inner iterations are applied to the term $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ at each time step to cope with the non-linearity caused by Z_m. Before the commencement of the

$$
\begin{array}{lll}
a_{n-1,n-4} & = & 1-EZ_m-M+K, \\
a_{n-1,n-3} & = & 120-28EZ_m-8M-4K, \\
a_{n-1,n-2} & = & \frac{353726}{297}-\frac{72766}{297}EZ_m+\frac{5642}{297}M+\frac{1486}{297}K, \\
a_{n-1,n-1} & = & \frac{717432}{297}-\frac{120}{297}EZ_m-\frac{120}{297}M+\frac{120}{297}K, \\
a_{n-1,n} & = & \frac{352833}{297}+\frac{71871}{297}EZ_m-\frac{6537}{297}M-\frac{591}{297}K, \\
a_{n-1,n+1} & = & \frac{34432}{297}+\frac{7108}{297}EZ_m+\frac{1168}{297}M+\frac{2396}{297}K, \\
a_{n,n-3} & = & 1-EZ_m-M+K, \\
a_{n,n-2} & = & \frac{35525.5}{297}-\frac{8338.5}{297}EZ_m-\frac{2378.5}{297}M-\frac{1197.5}{297}K, \\
a_{n,n-1} & = & \frac{340284}{297}-\frac{75168}{297}EZ_m+\frac{5640}{297}M+\frac{48}{297}K, \\
a_{n,n} & = & \frac{616822.5}{297}-\frac{18481.5}{297}EZ_m-\frac{601.5}{297}M-\frac{10126.5}{297}K, \\
a_{n,n+1} & = & \frac{215411}{297}+\frac{45585}{297}EZ_m-\frac{8663}{297}M-\frac{12961}{297}K, \\
a_{n+1,n-2} & = & -\frac{154}{297}EZ_m+\frac{90}{297}K, \\
a_{n+1,n-1} & = & -\frac{18480}{297}EZ_m+\frac{10800}{297}K, \\
a_{n+1,n} & = & -\frac{183414}{297}EZ_m+\frac{107190}{297}K, \\
a_{n+1,n+1} & = & -\frac{186032}{297}EZ_m+\frac{108720}{297}K,
\end{array}
$$

$$
\begin{array}{l}
b_{n-1,n-4}=1+EZ_m+M-K, \\
b_{n-1,n-3}=120+28EZ_m+8M+4K, \\
b_{n-1,n-2}=\frac{353726}{297}+\frac{72766}{297}EZ_m-\frac{5642}{297}M-\frac{1486}{297}K, \\
b_{n-1,n-1}=\frac{717432}{297}+\frac{120}{297}EZ_m+\frac{120}{297}M-\frac{120}{297}K, \\
b_{n-1,n}=\frac{352833}{297}-\frac{71871}{297}EZ_m+\frac{6537}{297}M+\frac{591}{297}K, \\
b_{n-1,n+1}=\frac{34432}{297}-\frac{7108}{297}EZ_m-\frac{1168}{297}M-\frac{2396}{297}K_3, \\
b_{n,n-3}=1+EZ_m+M-K, \\
b_{n,n-2}=\frac{35525.5}{297}+\frac{8338.5}{297}EZ_m+\frac{2378.5}{297}M+\frac{1197.5}{297}K, \\
b_{n,n-1}=\frac{340284}{297}+\frac{75168}{297}EZ_m-\frac{5640}{297}M-\frac{48}{297}K, \\
b_{n,n}=\frac{616822.5}{297}+\frac{18481.5}{297}EZ_m+\frac{601.5}{297}M+\frac{10126.5}{297}K, \\
b_{n,n+1}=\frac{215411}{297}-\frac{45585}{297}EZ_m+\frac{8663}{297}M+\frac{12961}{297}K, \\
b_{n+1,n-2}=\frac{154}{297}EZ_m-\frac{90}{297}K, \\
b_{n+1,n-1}=\frac{18480}{297}EZ_m-\frac{10800}{297}K, \\
b_{n+1,n}=\frac{183414}{297}EZ_m-\frac{107190}{297}K, \\
b_{n+1,n+1}=\frac{186032}{297}EZ_m-\frac{108720}{297}K.
\end{array}
$$

solution process, initial parameters d^0 must be determined by using the initial condition and following derivatives at the boundaries;

$$
\begin{array}{rcll}
U_N(x,0) & = & U(x_m,0) & m=0,1,2,...,N \\
(U_N)_x(a,0) & = & 0, & (U_N)_x(b,0)=0, \\
(U_N)_{xx}(a,0) & = & 0, & (U_N)_{xx}(b,0)=0, \\
(U_N)_{xxx}(a,0) & = & 0, & (U_N)_{xxx}(b,0)=0.
\end{array}
$$

So we have the following matrix form for the initial vector d^0;

$$Wd^0=b,$$

where
$d^0=(\delta_0,\delta_1,\delta_2,...,\delta_{N-2},\delta_{N-1},\delta_N)^T$
$b=(U(x_0,0),U(x_1,0),...,U(x_{N-1},0),U(x_N,0))^T$.

3. A linear stability analysis

To apply the Von-Neumann stability analysis, the Kawahara equation can be linearized by assuming that the quantity U in the nonlinear term UU_x is locally constant. Substituting the Fourier mode $\delta_j^n=\hat{\delta}^n e^{ijkh}$ $(i=\sqrt{-1})$ in which k is a mode number and h is the element size, into the Eq.(10) gives the growth factor g of the form

$$g=-\frac{a+ib}{a-ib},$$

$$W = \begin{bmatrix} 1536 & 2712 & 768 & 24 & & & & & & & \\ \frac{82731}{81} & \frac{210568.5}{81} & \frac{104796}{81} & \frac{10063.5}{81} & 1 & & & & & & \\ \frac{9600}{81} & \frac{96597}{81} & \frac{195768}{81} & \frac{96474}{81} & 120 & 1 & & & & & \\ & & \ddots & & & & & & & & \\ & & & & & & & & & & \\ & & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ & & & & 1 & 120 & \frac{96474}{81} & \frac{195768}{81} & \frac{96597}{81} & \frac{9600}{81} \\ & & & & & 1 & \frac{10063.5}{81} & \frac{104796}{81} & \frac{210568.5}{81} & \frac{82731}{81} \\ & & & & & & 24 & 768 & 2712 & 1536 \end{bmatrix}$$

where

$$\begin{aligned} a &= (19M + 5T - 245EZ_m)\sin[hk] + (-8M - 4T - 56EZ_m)\sin[2hk] + \\ &\quad (-M + T - EZ_m)\sin[3hk], \\ b &= -1208 - 1191\cos[hk] - 120\cos[2hk] - \cos[3hk]. \end{aligned} \tag{13}$$

The modulus of $|g|$ is 1, therefore the linearized scheme is unconditionally stable.

4. Numerical examples and results

To obtain the approximate solutions of the Eq.(2), we consider some numerical experiments including: motion of single solitary wave and interaction of two solitary waves. To show how good the numerical scheme predicts the position and amplitude of the solution as the simulation proceeds, we use the error norm L_2

$$L_2 = \left\| U^{exact} - U_N \right\|_2 \simeq \sqrt{h \sum_{J=1}^{N} \left| U_j^{exact} - (U_N)_j \right|^2},$$

and the error norm L_∞

$$L_\infty = \left\| U^{exact} - U_N \right\|_\infty \simeq \max_j \left| U_j^{exact} - (U_N)_j \right|, \quad j = 1, 2, ..., N-1.$$

The equation possesses the lowest three conserved quantities given by[12]

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} U dx \simeq h \sum_{i=1}^{N} U_i^n, \\ I_2 &= \int_{-\infty}^{\infty} \tfrac{1}{2} U^2 dx \simeq h \sum_{i=1}^{N} \tfrac{1}{2} (U_i^n)^2, \\ I_3 &= \int_{-\infty}^{\infty} (\tfrac{13}{8} \beta U_x^2 - \tfrac{105}{96} \alpha^2 U^3 + \tfrac{1}{2} U_{xx}^2) dx \simeq \\ &h \sum_{i=1}^{N} \tfrac{13}{8} \beta ((U_x)_i^n)^2 - \tfrac{105}{96} \alpha^2 (U_i^n)^3 + \tfrac{1}{2} ((U_{xx})_i^n)^2 \end{aligned} \tag{14}$$

We can calculate the integrals (14) by using the trapezium rule and the variables U_i^n and their derivatives are obtained from Eq.(7). To show the efficient of the numerical scheme, the above conserved quantities are expected to remain constant during the run of the algorithm so conserved quantities will be monitored.

4.1. The motion of single solitary wave

For this problem, Eq.(2) has a solitary wave solution of the form

$$U(x,t) = (\frac{\beta}{\alpha})^2 \sec h^4 \left[\frac{1}{4}\sqrt{\beta}\left(x - \frac{36}{169}t - x_0\right)\right]$$

where $(\frac{\beta}{\alpha})^2$ is the amplitude of the single solitary wave[13]. It represents a single solitary wave initially centred on x_0 and moving to the right with velocity $V = (1.5\beta)^2$. Initial condition

$$U(x,0) = (\frac{\beta}{\alpha})^2 \sec h^4 \left[\frac{1}{4}\sqrt{\beta}\,(x - x_0)\right],$$

and the boundary conditions $\alpha_1 = 0$, $\alpha_2 = 0$ are taken at the boundaries.

The values of the parameters $h = 0.2$, $\Delta t = 0.001$, $x_0 = 2$, $\alpha = 4/\sqrt{105}$, $\beta = 4/13$ are chosen over the spatial interval $[-20, 30]$ to coincide with that of previous papers[3-5]. For these parameters, the single solitary wave has an amplitude $A = (\frac{\beta}{\alpha})^2 = 0.6213$ and velocity $V = (1.5\beta)^2 = 0.2130$. Numerical computations are done up to time $t = 25$ to obtain the invariants and error norms L_2 and L_∞ at various times. Error norms, three conserved quantities, amplitudes and peak position of the solitary waves are listed in Table(I). It is clearly seen from the table that the invariants remain almost constant during the computer run. Table(II) shows a comparison of the values of the invariants and error norms obtained by the present method with obtained by the methods[3-5]. It is observed from the table that the error norms obtained by our method is smaller than given in Ref.[3] and almost the same in Refs.[4, 5]. Values of the invariants are also found in good agreement with the others. Solitary wave profiles with $h = 0.2$, $\Delta t = 0.001$ are depicted at different time levels in Fig.(1). As it is seen from the figure, the soliton moves to the right at a constant speed and almost unchanged amplitude as time increases, as expected.

Table I. The invariants and the error norms for single solitary wave with $h = 0.2$, $\Delta t = 0.001$, $-20 \le x \le 30$.

t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$	Amp.	Peak Position
0	5.9736088	1.2725033	-0.1645840	0.0000	0.0000	6.2130	2
5	5.9738529	1.2725034	-0.1620645	0.3249	0.1116	6.2120	3
10	5.9738517	1.2725033	-0.1633093	0.2469	0.0852	6.2118	4.2
15	5.9737831	1.2725033	-0.1640871	0.1807	0.0744	6.2130	5.2
20	5.9737261	1.2725033	-0.1643599	0.1529	0.0548	6.2121	6.2
25	5.9736181	1.2725033	-0.1644699	0.1395	0.0511	6.2116	7.4

4.2. Interaction of two solitary waves

In this section, we consider Eq.(2) with the boundary conditions $U(-50) = U(100) = 0$, by using the following initial condition

Table II. The errors and the invariants for single solitary wave with $h = 0.2$,$\Delta t = 0.001$, $-20 \leq x \leq 30$, at $t = 25$.

Method	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Present	5.97361	1.27250	-0.16446	0.139	0.051
[3](PDQ)	5.97353	1.27250	-0.16457	2.851	0.863
[3](CDQ)	5.97350	1.27250	-0.16458	0.159	0.076
[4]	5.97367	1.27250	-0.16459	0.093	0.023
[5](GA)	5.97353	1.27250	-0.16458	0.131	0.039
[5](MQ)	5.97355	1.27250	-0.16458	0.168	0.046

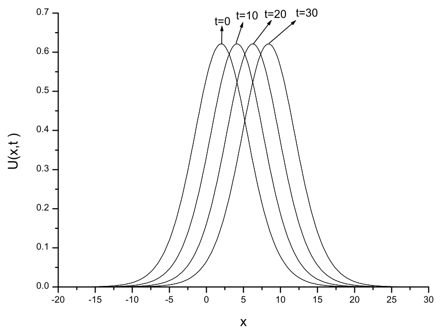

Figure 1. Single solitary wave with $h = 0.2$, $\Delta t = 0.001$, $-20 \leq x \leq 30$, $t = 0, 10, 20$ and 30.

$$U(x,0) = \sum_{i=1}^{2} A_i^2 \sec h^4\left(\frac{\sqrt{\alpha A_i}}{4}[x - x_i]\right), \tag{15}$$

where $\alpha = 4/\sqrt{105}$, $A_i = \beta_i/\alpha$, $\beta_i = (12 - 2i)/13$, $\Delta t = 0.01$, $h = 0.2$. Eq.(15) represents two solitary waves having different amplitudes, one with amplitude A_1 placed initially at $x_1 = 0$ and the second with amplitude A_2 placed at $x_2 = 20$.

A comparison of the values of the invariants obtained by the present method with those obtained in Refs.[3-5] given in Table(III). We notice that the obtained values of the invariants remain almost constant during the computer run. They are also found to be very close with the other earlier papers. Fig.(2) shows the interaction of solitary waves. As it is seen from the figure, at $t = 0$, a wave with larger amplitude is on the left of the another wave with smaller amplitude. The larger wave catches up with the smaller one as the time increases. Interaction started at about time $t = 20$, overlapping processes occurred between times $t = 20$ and $t = 40$ and, waves started to resume their original shapes after time $t = 40$. At $t = 50$, the amplitude of larger waves is 4.36993 at the point $x = 83.4$ whereas the amplitude of the smaller one is 2.75247 at the point $x = 62.8$. It is found that the absolute difference in amplitude is 2.67×10^{-1} for the smaller wave and 4.86×10^{-1} for the larger wave. Also, oscillations of small amplitude trailing behind the solitary waves were observed in the Figure.

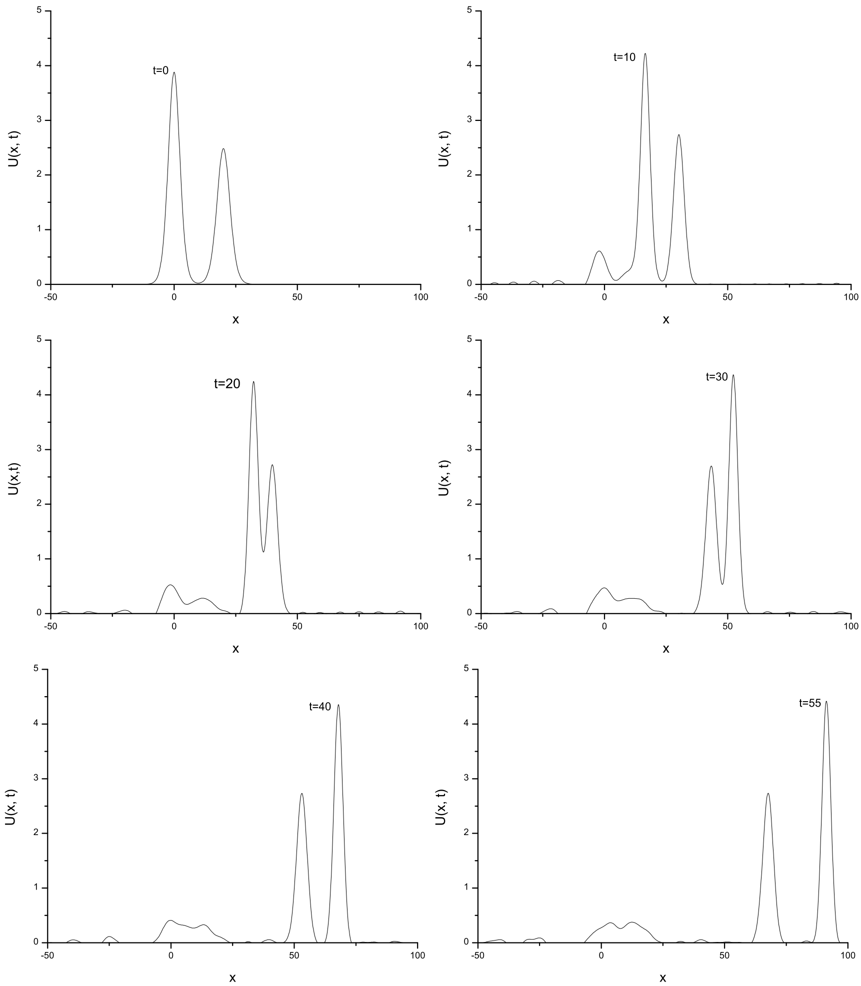

Figure 2. Interaction of two solitary waves with $t = 0,\ 10,\ 20,\ 30,\ 40,\ 55.$

5. Conclusion

In this paper, a numerical scheme based on the septic B-spline collocation method have been set up to find numerical solution of the Kawahara equation. Unconditional stability is shown by using Von-Neumann analysis. To show the performance of the method, we have examined the motion of a solitary wave and the interaction of two solitary waves. Efficiency and accuracy of the method is shown by calculating L_2 , L_∞ error norms and I_1, I_2 and I_3 invariant quantities. The obtained results show that the error norms are adequately small and the conservation laws are reasonably well satisfied for

Table III. Comparison of the invariants for the interaction of two solitary waves with the parameters $h = 0.2$, $\Delta t = 0.01$ in the interval $-50 \leq x \leq 100$.

Present method			
t	I_1	I_2	I_3
0	40.50920	45.83608	-37.87770
10	40.45628	45.83603	-34.73750
20	40.46943	45.83597	-34.97900
30	40.55136	45.83593	-33.83404
40	40.45048	45.83586	-33.69678
50	40.34304	45.83577	-33.17479
55[4]	4040483	45.83524	-32.37120
50[5](GA)	40.41284	45.84364	-32.14082
50[5](MQ)	40.48389	45.85093	-32.15991

the interaction of two solitary waves. Thus, this numerical algorithm can be used reliably to obtain numerical solutions of the differential equations.

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