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Existence of positive solutions to multi-point third order problems with sign changing nonlinearities

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Abstract. In this paper, the authors examine the existence of positive solutions to a third-order boundary value problem having a sign changing nonlinearity. The proof makes use of fixed point index theory. An example is included to illustrate the applicability of the results.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics such as the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves, gravity driven flows, etc. (see [10]). Anderson [1] proved the existence of solutions to the third-order nonlinear boundary value problem (BVP)

$$x'''(t) = f(t, x(t)), \quad t_1 \le t \le t_3,$$
$$x(t_1) = x'(t_2) = 0, \quad \gamma x(t_3) + \delta x''(t_3) = 0,$$

by applying the Krasnosel'skii and the Leggett and Williams fixed point theorems. Dogan [5] investigated the existence of positive solutions to the multi-point problem

$$u'' + \lambda f(t, u) = 0, \quad 0 < t < 1,$$
$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i),$$

where $\xi_i \in (0, 1)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\alpha_i\, \beta_i \in [0, \infty)$, and λ is a positive parameter. He used Krasnosel'skii's fixed point theorem to give

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sufficient conditions for the existence of at least one positive solution to this problem. Emphasis was put on the fact that the nonlinear term f could take negative values.

Graef and Kong [8] studied the BVP consisting of a forced nonlinear third order differential equation together with multi-point boundary conditions (BC)

$$\begin{aligned} &u'''(t) = \lambda f(t, u(t)) + e(t), \qquad 0 < t < 1, \\ &u(0) = u'(p) = \int_q^1 w(s)u''(s)ds = 0, \end{aligned}$$

where $\lambda > 0$ is a parameter, $1/2 < p < q < 1$ are constants, $f : (0,1) \times [0,\infty) \to \mathbb{R}$, $e : (0,1) \to \mathbb{R}$, and $w : [q,1] \to [0,\infty)$ are continuous functions with $e \in L(0,1)$. They gave sufficient conditions for the existence of positive solutions.

Henderson and Kosmatov [12] considered the third-order nonlinear BVP

$$\begin{gathered} u'''(t) - f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1/2) = u''(1) = 0, \end{gathered}$$

with a sign-changing nonlinearity. Li [15] studied the existence of single and multiple positive solutions to the nonlinear singular third-order two-point problem

$$\begin{gathered} u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0. \end{gathered}$$

Using Krasnosel'skii's fixed-point theorem, he established intervals on the parameter λ that yield the existence of at least one, at least two, and infinitely many positive solutions of the BVP.

Liu et al. [16] considered the existence of positive solutions to the third order two-point generalized right focal BVP

$$\begin{gathered} x'''(t) + f(t, x(t)) = 0, \quad a < t < b, \\ x(a) = x'(a) = x''(b) = 0 \end{gathered}$$

and gave some new results for the existence of at least one, two, three, and infinitely many monotone positive solutions. They also used Krasnosel'skii's fixed point theorem and the Leggett-Williams fixed-point theorem.

Sun and Zhao [22] considered the third order three-point problem

$$\begin{gathered} u'''(t) = f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u''(\eta) = u(1) = 0, \end{gathered}$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$ and $\eta \in [2 - \sqrt{2}, 1)$. Although the corresponding Green's function is sign-changing, they found the existence of a decreasing positive solution under some suitable conditions on f. Their technique of proof involved using an iterative approach.

The third-order Sturm-Liouville BVP with a p-Laplacian

$$(\phi_p(u''(t)))' + f(t,u(t)) = 0, \quad t \in (0,1),$$
$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \quad u''(0) = 0,$$

was studied by Yang and Yan [26]. They proved the existence of at least one or at least two positive solutions by using fixed point index theory.

Zhang et al. [27] studied the existence and uniqueness of nontrivial solutions of the third-order eigenvalue problem

$$u''' = \lambda f(t,u,u'), \quad 0 < t < 1,$$
$$u(0) = u'(\eta) = u''(0) = 0.$$

Without requiring any monotone-type or nonnegative assumptions, they found several sufficient conditions for the existence and uniqueness of nontrivial solution for λ in certain intervals. Their approach was based on the Leray-Schauder nonlinear alternative. Zhou and Ma [28] showed the existence of positive solutions and established a corresponding iterative scheme for the third-order generalized right-focal BVP

$$(\phi_p(u''))' = q(t)f(t,u(t)), \quad 0 \le t \le 1,$$
$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \quad u'(\eta) = 0, \quad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i)$$

by using the monotone iterative technique.

In [21], Sang and Su considered the problem

$$(\phi(u''))' + a(t)f(t,u(t)) = 0, \quad 0 < t < 1,$$
$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i \phi(u''(\xi_i)), \quad u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

and gave sufficient conditions for the existence of one or two positive solutions.

While recently the question of the existence of positive solutions to nonlinear BVPs has been studied extensively in the literature, (see, for example, [1, 2, 3, 4, 5, 6, 7, 9, 13, 16, 17, 15, 18, 19, 20, 23, 22, 25, 24, 28]), there are relatively few results on the existence of positive solutions for third order multi-point BVPs with sign changing nonlinearities (see [21]). Our purpose here then is to consider the multi-point nonlinear third order problem

$$(\phi(u''))' + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u''(0)) = \sum_{i=1}^{m-2} b_i \phi(u''(\xi_i)), \quad u'(1) = 0, \tag{2}$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism and homomorphism with $\phi(0) = 0$.

A projection $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is called an *increasing homeomorphism and homomorphism* if the following conditions are satisfied:

(i) if $x \leq y$, then $\phi(x) \leq \phi(y)$ for all $x, y \in \mathbb{R}$;
(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$.

Throughout the paper, we assume that the following conditions are satisfied:

(H1) $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and a_i, $b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, $0 < \sum_{i=1}^{m-2} b_i < 1$, and $0 < \sum_{i=1}^{m-2} a_i(1-\xi_i) < 1$;
(H2) $q : [0,1] \to [0,+\infty)$ is continuous and there exists $t_0 \in (0,1)$ such that $q(t_0) > 0$;
(H3) $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ is continuous with $f(t,0) \geq 0$.

Motivated by the work described above, our aim is to give some existence results for positive solutions to BVP (1)–(2). To the best of our knowledge, there are still no results for the existence of positive solutions to BVP (1)–(2) by using fixed point index theory. It is important to point out that the nonlinear term f may change sign.

This paper is organized as follows. In Section 2, we present some lemmas that will be used to prove our main results. In Section 3, we present the main results in our paper; they generalize and extend corresponding works in the references. At the end of the paper an example is included to illustrate the results.

2. Preliminaries

To find positive solutions of BVP (1)–(2), the following fixed point theorem in cones is fundamental.

Theorem 2.1 (see [11]). *Let K be a cone in a Banach space X and D be an bounded open subset of X with $D_K = D \cap K \neq \varnothing$ and $\overline{D_K} \neq K$. Let $F : \overline{D_K} \to K$ be a completely continuous map such that $u \neq Fu$ for $u \in \partial D_K$. Let $i(F, D_K, K)$ denote a fixed point index. Then the following results are satisfied.*

(A1) *If $\|Fu\| \leq \|u\|$, $u \in \partial D_K$, then $i(F, D_K, K) = 1$.*
(A2) *If there exists $e \in K \setminus \{0\}$ such that $u \neq Fu + \lambda e$, $u \in \partial D_K$, and $\lambda > 0$, then $i(F, D_K, K) = 0$.*
(A3) *Let U be open in X such that $\overline{U} \subset D_K$. If $i(F, D_K, K) = 1$ and $i(F, U_K, K) = 0$, then F has a fixed point in $D_K \setminus \overline{U_K}$. The same result holds if $i(F, D_K, K) = 0$ and $i(F, U_K, K) = 1$.*

The following lemmas will play important roles in our proofs.

Lemma 2.1. *Let* $1-\sum_{i=1}^{m-2} a_i \neq 0$, $1-\sum_{i=1}^{m-2} b_i \neq 0$, *and* $h \in C[0,1]$. *Then* $u(t)$ *is the unique solution of the BVP*

$$(\phi(u''))' + h(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \phi(u''(0)) = \sum_{i=1}^{m-2} b_i \phi(u''(\xi_i)), \qquad u'(1) = 0, \tag{4}$$

if and only if

$$u(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2 t + C_3, \tag{5}$$

where

$$\begin{aligned}
C_1 &= -\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1-\sum_{i=1}^{m-2} b_i},\\
C_2 &= -\int_0^1 \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds,\\
C_3 &= \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds}{1-\sum_{i=1}^{m-2} a_i}\\
&\quad - \frac{\sum_{i=1}^{m-2} a_i \xi_i \int_0^1 \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds}{1-\sum_{i=1}^{m-2} a_i}.
\end{aligned}$$

Proof. First, we prove the necessity. Integrating the differential equation from 0 to t gives

$$\phi(u''(t)) = -\int_0^t h(r)dr + C_1, \tag{6}$$

i.e.,

$$u''(t) = \phi^{-1}\left(-\int_0^t h(r)dr + C_1\right).$$

Integrating from 0 to t, we have

$$u'(t) = \int_0^t \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2. \tag{7}$$

A final integration yields

$$u(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2 t + C_3. \tag{8}$$

Setting $t = 0$ and $t = \xi_i$ in (6) gives $\phi(u''(0)) = C_1$ and

$$\phi(u''(\xi_i)) = -\int_0^{\xi_i} h(r)dr + C_1.$$

Setting $t = 1$ in (7), we have

$$u'(1) = \int_0^1 \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2.$$

With $t = 0$, (8) becomes

$$u(0) = C_3.$$

Similarly, we have

$$u(\xi_i) = \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2\xi_i + C_3.$$

Applying BC (4) gives

$$C_1 = -\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i},$$

$$C_2 = -\int_0^1 \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds,$$

$$C_3 = \sum_{i=1}^{m-2} a_i \left[\int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2\xi_i + C_3\right],$$

so

$$C_3 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds}{1 - \sum_{i=1}^{m-2} a_i} - \frac{\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds}{1 - \sum_{i=1}^{m-2} a_i}.$$

To prove sufficiency, let u be as in (5). Then

$$u'(t) = \int_0^t \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2,$$

$$u''(t) = \phi^{-1}\left(-\int_0^t h(r)dr + C_1\right),$$

and

$$\phi(u''(t)) = -\int_0^t h(r)dr + C_1.$$

Differentiating again, we obtain $(\phi(u''))' = -h(t)$. Standard calculations verify that u satisfies the BCs in (4), so that u given in (5) is a solution of BVP (3)–(4). It can be readily seen that the BVP

$$(\phi(u''))' = 0, \quad u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u''(0)) = \sum_{i=1}^{m-2} b_i \phi(u''(\xi_i)), \quad u'(1) = 0 \tag{9}$$

has only the trivial solution if

$$1-\sum_{i=1}^{m-2} a_i \neq 0 \quad \text{and} \quad 1-\sum_{i=1}^{m-2} b_i \neq 0.$$

Thus, u in (5) is the unique solution of BVP (3)–(4), and this completes the proof of the lemma. □

Lemma 2.2. *Let (H1) hold and $h \in C[0,1]$ with $h(t) \geq 0$. Then, the unique solution u of BVP (3)–(4) satisfies $u(t) \geq 0$ on $[0,1]$. Moreover,*

$$\inf_{t\in[0,1]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i(1-\xi_i)} \quad \textit{and} \quad \|u\| = \max_{t\in[0,1]} |u(t)|.$$

Proof. Let

$$\varphi_0(s) = \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right).$$

Since

$$-\int_0^s h(r)dr + C_1 = -\int_0^s h(r)dr - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1-\sum_{i=1}^{m-2} b_i} \leq 0,$$

it follows that $\varphi_0(s) \leq 0$. For $0 < t \leq 1$,

$$\left(\frac{\int_0^t (t-s)\varphi_0(s)ds}{t}\right)' = \frac{t\int_0^t \varphi_0(s)ds - \int_0^t (t-s)\varphi_0(s)ds}{t^2} \leq 0.$$

Now

$$\frac{\int_0^t (t-s)\varphi_0(s)ds}{t} \geq \frac{\int_0^1 (1-s)\varphi_0(s)ds}{1}$$

for $0 < t \leq 1$, so for ξ_i, $i = 1, 2, \ldots, m-2$, we see that

$$\int_0^{\xi_i} (\xi_i - s)\varphi_0(s)ds \geq \frac{\xi_i}{1}\int_0^1 (1-s)\varphi_0(s)ds. \tag{10}$$

Using Lemma 2.1 and (10), we obtain

$$\begin{aligned}
u(0) &= \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \\
&\geq \frac{\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 (1-s)\varphi_0(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \\
&= -\frac{\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 s\varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \geq 0
\end{aligned}$$

and

$$
\begin{aligned}
u(1) &= \int_0^1 (1-s)\varphi_0(s)ds - \int_0^1 \varphi_0(s)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \\
&\geq \int_0^1 (1-s)\varphi_0(s)ds - \int_0^1 \varphi_0(s)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 (1-s)\varphi_0(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \\
&= -\int_0^1 s\varphi_0(s)ds - \frac{\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 s\varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \geq 0.
\end{aligned}
$$

If $t \in (0,1)$, we have

$$
\begin{aligned}
u(t) &= \int_0^t (t-s)\varphi_0(s)ds - t\int_0^1 \varphi_0(s)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \\
&\geq \int_0^1 (1-s)\varphi_0(s)ds - \int_0^1 \varphi_0(s)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \\
&\geq \int_0^1 (1-s)\varphi_0(s)ds - \int_0^1 \varphi_0(s)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 (1-s)\varphi_0(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \\
&= -\int_0^1 s\varphi_0(s)ds - \frac{\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi_0(s)ds}{1-\sum_{i=1}^{m-2} a_i} \geq 0.
\end{aligned}
$$

Hence, $u(t) \geq 0$ for $t \in [0,1]$.

Now taking the derivative of (5) with respect to t, we obtain

$$
\begin{aligned}
u'(t) &= \int_0^t \phi^{-1}\left(-\int_0^s h(r)dr + C_1\right) ds + C_2 \\
&= \int_0^t \phi^{-1}\left(-\int_0^s h(r)dr - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1-\sum_{i=1}^{m-2} b_i}\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_0^1 \phi^{-1}\left(- \int_0^s h(r)dr - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \\
&= - \int_0^t \phi^{-1}\left(\int_0^s h(r)dr + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \\
&\quad + \int_0^1 \phi^{-1}\left(\int_0^s h(r)dr + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \\
&= - \int_0^t \phi^{-1}\left(\int_0^s h(r)dr + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \\
&\quad + \left[\int_0^t \phi^{-1}\left(\int_0^s h(r)dr + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \right. \\
&\quad \left. + \int_t^1 \phi^{-1}\left(\int_0^s h(r)dr + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \right] \\
&= \int_t^1 \phi^{-1}\left(\int_0^s h(r)dr + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(r)dr}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \geq 0.
\end{aligned}
$$

This implies

$$\min_{t\in[0,1]} u(t) = u(0) \quad \text{and} \quad \|u\| = u(1).$$

It is easy to see that $u'(t_2) \leq u'(t_1)$ for any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$. Hence, $u'(t)$ is decreasing on [0,1], so the graph of $u(t)$ is concave down on (0,1). For each $i \in \{1, 2, \dots, m-2\}$, we have

$$\frac{u(1) - u(0)}{1 - 0} \geq \frac{u(1) - u(\xi_i)}{1 - \xi_i},$$

i.e.,

$$u(\xi_i) - \xi_i u(1) \geq (1 - \xi_i)u(0),$$

so that

$$\sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i \xi_i u(1) \geq \sum_{i=1}^{m-2} a_i (1 - \xi_i) u(0).$$

Applying the BC $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, we have

$$u(0) \geq \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i (1 - \xi_i)} u(1).$$

This completes the proof of the lemma. □

3. Main results

In order to present our main results, we first define our Banach space, an appropriate cone, and an operator. Let $E = C[0,1]$; then E is a Banach space with respect to the norm $\|\cdot\|$ defined above. We take

$$K = \{u \in C[0,1] : u(t) \geq 0, \inf_{t\in[0,1]} u(t) \geq \gamma\|u\|\},$$

where γ is given in Lemma 2.2. It is easy to see that K is a cone in $C[0,1]$. For any constant $\rho > 0$, we define:

$$\begin{aligned} \varphi(t) &= \min\{t, 1-t\},\ t \in (0,1); \\ K_\rho &= \{u \in K : \|u\| < \rho\}; \\ K_\rho^* &= \{u \in K : \rho\varphi(t) < u(t) < \rho,\ t \in [0,1]\}; \\ \Omega_\rho &= \{u \in K : \min_{t\in[0,1]} u(t) < \gamma\rho\}. \end{aligned} \tag{11}$$

Lemma 3.1 (see [14])**.** *The set Ω_ρ has the following properties:*

(i) *Ω_ρ is open relative to K;*
(ii) *$K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$;*
(iii) *$u \in \partial\Omega_\rho$ if and only if $\min_{t\in[0,1]} u(t) = \gamma\rho$;*
(iv) *If $u \in \partial\Omega_\rho$, then $\gamma\rho \leq u(t) \leq \rho$ for $t \in [0,1]$.*

For convenience, we set

$$\begin{aligned} \frac{1}{\mu} &= \int_0^1 \phi^{-1}\left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right) ds \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right) ds, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \frac{1}{\delta} &= \int_0^1 s\phi^{-1}\left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right) ds \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 s\phi^{-1}\left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right) ds, \end{aligned} \tag{13}$$

where

$$\boldsymbol{\sigma}(q) = \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} q(r)dr.$$

Also, for $\alpha \in \{0^+, \infty\}$, we define

$$f^\alpha = \limsup_{u\to\alpha} \left\{\max_{t\in[0,1]} \frac{f(t,u)}{\phi(u)}\right\},$$

$$\begin{aligned}
f_\alpha &= \liminf_{u\to\alpha}\left\{\min_{t\in[0,1]}\frac{f(t,u)}{\phi(u)}\right\},\\
f^\rho_{\gamma\rho} &= \min\left\{\min_{t\in[0,1]}\frac{f(t,u)}{\phi(\rho)} : u\in[\gamma\rho,\rho]\right\},\\
f^\rho_{\varphi(t)\rho} &= \max\left\{\max_{t\in[0,1]}\frac{f(t,u)}{\phi(\rho)} : u\in[\varphi(t)\rho,\rho]\right\},\\
f^\rho_0 &= \max\left\{\max_{t\in[0,1]}\frac{f(t,u)}{\phi(\rho)} : u\in[0,\rho]\right\}.
\end{aligned}$$

Our main result in this paper is contained in the following theorem.

Theorem 3.1. *In addition to conditions(H1)–(H3), assume that one of the following conditions holds:*

(B1) *There exist ρ_1, $\rho_2\in(0,+\infty)$ with $\rho_1<\gamma\rho_2$ such that*
  (i) *$f(t,u(t))>0$ for $t\in[0,1]$ and $u(t)\in[\rho_1\varphi(t),+\infty)$, and*
  (ii) *$f^{\rho_1}_{\varphi(t)\rho_1}\le\phi(\mu)$ for $t\in[0,1]$ and $f^{\rho_2}_{\gamma\rho_2}\ge\phi(\delta\gamma)$;*

(B2) *There exist ρ_1, $\rho_2\in(0,+\infty)$ with $\rho_1<\rho_2$ such that*
  (i) *$f(t,u(t))>0$ for $t\in[0,1]$ and $u(t)\in[\min\{\gamma\rho_1,\rho_2\varphi(t)\},+\infty)$, and*
  (ii) *$f^{\rho_1}_{\gamma\rho_1}\ge\phi(\delta\gamma)$ and $f^{\rho_2}_{\varphi(t)\rho_2}\le\phi(\mu)$ for $t\in[0,1]$.*

Then BVP (1)–(2) *has at least one positive solution.*

In oder to prove this theorem, first note that if we define the function f^* by

$$f^*(t,u)=\begin{cases} f(t,u), & u\ge\rho_1\varphi(t),\\ f(t,\rho_1\varphi(t)), & 0\le u<\rho_1\varphi(t),\end{cases}$$

for $t\in[0,1]$, then it is easy to see that $f^*(t,u)\in C([0,1]\times[0,+\infty),(0,+\infty))$. Consider the following modified BVP:

$$(\phi(u''))'+q(t)f^*(t,u(t))=0,\qquad t\in(0,1),$$
$$u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i),\quad \phi(u''(0))=\sum_{i=1}^{m-2}b_i\phi(u''(\xi_i)),\quad u'(1)=0.$$

We define the operator $F:K\to E$ by

$$(Fu)(t)=\int_0^t(t-s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr+\tilde{C}_1\right)ds+\tilde{C}_2t+\tilde{C}_3,$$

where

$$\tilde{C}_1=-\frac{\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}q(r)f^*(r,u(r))dr}{1-\sum_{i=1}^{m-2}b_i},$$

$$\tilde{C_2} = -\int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr + \tilde{C_1}\right)ds,$$

$$\tilde{C_3} = \frac{1}{1-\sum_{i=1}^{m-2} a_i}\left[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr + \tilde{C_1}\right)ds \right.$$
$$\left. - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr + \tilde{C_1}\right)ds\right].$$

We will need the following lemma in the proof of our main result.

Lemma 3.2. *The function $F : K \to K$ is completely continuous.*

Proof. First, we show that $F(K) \subset K$. For each $u \in K$ it is easy to check that Fu is nonnegative, concave, and nondecreasing on $[0, 1]$. Moreover, it is clear that Fu satisfies (4). Hence, Lemma 2.2 implies that the Harnack type inequality

$$\inf_{t\in[0,1]} (Fu)(t) \geq \gamma\|Fu\|$$

holds for $u \in K$. Thus, $F(K) \subset K$.

Next, we prove that F maps a bounded set into itself. Let $c > 0$ be a constant and $u \in \overline{K_c} = \{u \in K : \|u\| \leq c\}$. Note that the continuity of f^* guarantees that there is a $L > 0$ such that $f^*(t, u(t)) \leq \phi(L)$ for $t \in [0, 1]$. Therefore, using also the notation

$$\boldsymbol{\sigma}(q,u) = \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} q(r)f^*(r,u(r))dr,$$

we get

$$\begin{aligned}
\|Fu\| &= \max_{t\in[0,1]} Fu(t) \\
&= \int_0^1 (1-s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr + \tilde{C_1}\right)ds + \tilde{C_2} + \tilde{C_3} \\
&= \int_0^1 (1-s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right)ds \\
&\quad - \int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right)ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\left[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\Bigg(-\int_0^s q(r)f^*(r,u(r))dr\right. \\
&\quad - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\Bigg)ds
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right) ds\Bigg] \\
&= -\int_0^1 \phi^{-1}\left(\int_0^s q(r)f^*(r,u(r))dr + \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right) ds \\
&+ \int_0^1 s\phi^{-1}\left(\int_0^s q(r)f^*(r,u(r))dr + \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right) ds \\
&+ \int_0^1 \phi^{-1}\left(\int_0^s q(r)f^*(r,u(r))dr + \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right) ds \\
&+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)\phi^{-1}\Bigg(\int_0^s q(r)f^*(r,u(r))dr \\
&+ \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\Bigg)ds \\
&+ \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\left(\int_0^s q(r)f^*(r,u(r))dr + \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right) ds\Bigg] \\
&\le \int_0^1 s\phi^{-1}\left(\int_0^s q(r)f^*(r,u(r))dr + \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right) ds \\
&+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\Bigg(\int_0^s q(r)f^*(r,u(r))dr \\
&+ \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\Bigg)ds \\
&\le L\Bigg[\int_0^1 s\phi^{-1}\Bigg(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\Bigg)ds \\
&+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\Bigg(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} a_i}\Bigg)ds\Bigg].
\end{aligned}
$$

That is, $F\overline{K_c}$ is uniformly bounded.

Next, for any t_1, $t_2 \in [0,1]$, we have

$$
\begin{aligned}
\Big|Fu(t_1) - Fu(t_2)\Big| &= \Bigg|\int_0^{t_1}(t_1 - s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr + \tilde{C}_1\right) ds + \tilde{C}_2 t_1 \\
&- \int_0^{t_2}(t_2 - s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr + \tilde{C}_1\right) ds - \tilde{C}_2 t_2\Bigg|
\end{aligned}
$$

$$= \Bigg| \int_0^{t_1} (t_1 - s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$
$$- t_1 \int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$
$$- \int_0^{t_2} (t_2 - s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$
$$+ t_2 \int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right)ds\Bigg|$$
$$\leq L|t_1 - t_2|\left[\int_0^1 \phi^{-1}\left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right)ds\right] \to 0 \text{ as } t_1 \to t_2.$$

Therefore, by the Arzelà-Ascoli theorem, $F(K)$ is relatively compact.

Finally, we show that $F : \overline{K_c} \to K$ is continuous. Assume that $\{u_n\}_{n=1}^{\infty} \subset \overline{K_c}$ and $u_n(t)$ converges to $u_0(t)$ uniformly on $[0,1]$. Thus, $\{(Fu_n)(t)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0,1]$. By the Arzelà-Ascoli theorem, there is a uniformly convergent subsequence of $\{(Fu_n)(t)\}_{n=1}^{\infty}$, say $\{(Fu_{n(m)})(t)\}_{m=1}^{\infty}$, that converges to $v(t)$ uniformly on $[0,1]$. Observe that

$$(Fu_n)(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u_n(r))dr - \frac{\boldsymbol{\sigma}(q,u_n)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$
$$+ t\int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u_n(r))dr - \frac{\boldsymbol{\sigma}(q,u_n)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$
$$+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\left[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u_n(r))dr\right.\right.$$
$$\left. - \frac{\boldsymbol{\sigma}(q,u_n)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$
$$\left. - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u_n(r))dr - \frac{\boldsymbol{\sigma}(q,u)}{1-\sum_{i=1}^{m-2} b_i}\right)ds\right].$$

Along the subsequence $u_{n(m)}$ letting $m \to \infty$, we see that

$$v(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s q(r)f^*(r,u_0(r))dr - \frac{\boldsymbol{\sigma}(q,u_0)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$
$$+ t\int_0^1 \phi^{-1}\left(-\int_0^s q(r)f^*(r,u_0(r))dr - \frac{\boldsymbol{\sigma}(q,u_0)}{1-\sum_{i=1}^{m-2} b_i}\right)ds$$

$$
\begin{aligned}
&+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\Bigg(- \int_0^s q(r) f^*(r, u_0(r))dr \\
&- \frac{\boldsymbol{\sigma}(q, u_0)}{1-\sum_{i=1}^{m-2} b_i}\Bigg)ds \\
&- \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \phi^{-1}\Bigg(- \int_0^s q(r) f^*(r, u_0(r))dr - \frac{\boldsymbol{\sigma}(q, u_0)}{1-\sum_{i=1}^{m-2} b_i}\Bigg)ds\Bigg].
\end{aligned}
$$

From the definition of F, we know that $v(t) = Fu_0(t)$ on $[0, 1]$. This shows that each subsequence of $\{Fu_n(t)\}_{n=1}^{\infty}$ converges uniformly to $(Fu_0)(t)$. So the sequence $\{(Fu_n)(t)\}_{n=1}^{\infty}$ uniformly converges to $(Fu_0)(t)$. That is, F is continuous at $u_0 \in \overline{K_c}$. Since u_0 is arbitrary, F is completely continuous, and this proves the lemma. □

Proof of Theorem 3.1. Suppose that (B1) holds. We will show that the hypotheses of Theorem 2.1 are satisfied. First, we will show that

$$
i(F, K^*_{\rho_1}, K) = 1.
$$

Notice that from (12) and the facts that $f^{*\rho_1}_{\varphi(t)\rho_1} \le \phi(\mu)$ and $u \ne Fu$ for $u \in \partial K^*_{\rho_1}$, for every $u \in \partial K^*_{\rho_1}$, we have

$$
\begin{aligned}
- \int_0^s q(r) f^*(r, u(r))dr + \tilde{C}_1 &= - \int_0^s q(r) f^*(r, u(r))dr - \frac{\boldsymbol{\sigma}(q, u)}{1-\sum_{i=1}^{m-2} b_i} \\
&\ge -\phi(\rho_1)\phi(\mu)\left[\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right]
\end{aligned}
$$

so that

$$
\begin{aligned}
\varphi(s) &= \phi^{-1}\left(- \int_0^s q(r) f^*(r, u(r))dr + \tilde{C}_1\right) \\
&\ge -\rho_1\mu\phi^{-1}\left[\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|Fu\| &\le - \int_0^1 \varphi(s)ds + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad \times \left[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\varphi(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi(s)ds\right] \\
&\le - \int_0^1 \varphi(s)ds - \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \varphi(s)ds
\end{aligned}
$$

$$\leq \rho_1 \mu \Bigg[\int_0^1 \phi^{-1} \left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1 - \sum_{i=1}^{m-2} b_i} \right) ds$$
$$+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 \phi^{-1} \left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1 - \sum_{i=1}^{m-2} b_i} \right) ds \Bigg]$$
$$= \rho_1 = \|u\|.$$

This implies that $\|Fu\| \leq \|u\|$ for $u \in \partial K^*_{\rho_1}$. Hence, it follows from part (A1) of Theorem 2.1 that $i(F, K^*_{\rho_1}, K) = 1$.

Next, we show that

$$i(F, \Omega_{\rho_2}, K) = 0.$$

Let $e \equiv 1$ for $t \in [0, 1]$; then $e \in \partial K_1$. We claim that

$$u \neq Fu + \lambda e \quad \text{for} \quad u \in \partial \Omega_{\rho_2},\ \lambda > 0.$$

If this is not the case, then there exist $u_0 \in \partial \Omega_{\rho_2}$ and $\lambda_0 > 0$ such that

$$u_0 = Fu_0 + \lambda_0 e.$$

Then from (13) and the facts that $f^{*\rho_2}_{\gamma\rho_2} \geq \phi(\delta\gamma)$ and $u \neq Fu$ for $u \in \partial\Omega_{\rho_2}$, we obtain

$$-\int_0^s q(r) f^*(r, u_0(r))dr + \tilde{C}_1 \mid_{u=u_0}$$
$$= -\int_0^s q(r) f^*(r, u_0(r))dr - \frac{\boldsymbol{\sigma}(q, u_0)}{1 - \sum_{i=1}^{m-2} b_i}$$
$$\leq -\phi(\rho_2)\phi(\delta\gamma) \left[\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1 - \sum_{i=1}^{m-2} b_i} \right]$$

so that

$$\tilde{\varphi}(s) = \phi^{-1} \left(-\int_0^s q(r) f^*(r, u_0(r))dr + \tilde{C}_1 \mid_{u=u_0} \right)$$
$$\leq -\rho_2 \delta\gamma \phi^{-1} \left[\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1 - \sum_{i=1}^{m-2} b_i} \right].$$

For ξ_i, $i = 1, 2, \ldots, m-2$,

$$\int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)ds \geq \xi_i \int_0^1 (1 - s)\tilde{\varphi}(s)ds.$$

Since

$$-\int_0^s q(r) f^*(r, u_0(r))dr + \tilde{C}_1 \mid_{u=u_0} \;=\; -\int_0^s q(r) f^*(r, u_0(r))dr - \frac{\boldsymbol{\sigma}(q, u_0)}{1 - \sum_{i=1}^{m-2} b_i} \leq 0,$$

it follows that $\tilde{\varphi}(s) \le 0$. For $0 < t \le 1$,

$$\left(\frac{\int_0^t (t-s)\tilde{\varphi}(s)ds}{t}\right)' = \frac{t\int_0^t \tilde{\varphi}(s)ds - \int_0^t (t-s)\tilde{\varphi}(s)ds}{t^2} \le 0,$$

so

$$\frac{\int_0^t (t-s)\tilde{\varphi}(s)ds}{t} \ge \frac{\int_0^1 (1-s)\tilde{\varphi}(s)ds}{1}. \tag{14}$$

From (14), for ξ_i, $i = 1, 2, 3, \dots, m-2$, we have (see (10))

$$\int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)ds \ge \frac{\xi_i}{1}\int_0^1 (1-s)\tilde{\varphi}(s)ds. \tag{15}$$

Using (15), we see that

$$\begin{aligned}
u_0(t) &= Fu_0(t) + \lambda_0 e(t) \\
&\ge \int_0^1 (1-s)\tilde{\varphi}(s)ds - \int_0^1 \tilde{\varphi}(s)ds + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad \times \left(\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \tilde{\varphi}(s)ds\right) + \lambda_0 \\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[\int_0^1 (1-s)\tilde{\varphi}(s)ds - \sum_{i=1}^{m-2} a_i \int_0^1 (1-s)\tilde{\varphi}(s)ds \\
&\quad - \int_0^1 \tilde{\varphi}(s)ds + \sum_{i=1}^{m-2} a_i \int_0^1 \tilde{\varphi}(s)ds + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)ds \\
&\quad - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \tilde{\varphi}(s)ds\Bigg] + \lambda_0 \\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[-\int_0^1 s\tilde{\varphi}(s)ds + \sum_{i=1}^{m-2} a_i \int_0^1 s\tilde{\varphi}(s)ds \\
&\quad + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \tilde{\varphi}(s)ds\Bigg] + \lambda_0 \\
&= \quad \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[-\int_0^1 s\tilde{\varphi}(s)ds + \sum_{i=1}^{m-2} a_i \int_0^1 s\tilde{\varphi}(s)ds \\
&\quad + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\tilde{\varphi}(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \tilde{\varphi}(s)ds\Bigg] + \lambda_0 \\
&\ge \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[-\int_0^1 s\tilde{\varphi}(s)ds + \sum_{i=1}^{m-2} a_i \int_0^1 s\tilde{\varphi}(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 (1-s)\tilde{\varphi}(s)ds - \sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \tilde{\varphi}(s)ds\Bigg] + \lambda_0 \\
&= -\int_0^1 s\tilde{\varphi}(s)ds + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 (1-s)\tilde{\varphi}(s)ds \\
&\quad -\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 \tilde{\varphi}(s)ds\Bigg] + \lambda_0 \\
&= -\int_0^1 s\tilde{\varphi}(s)ds - \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Bigg[\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 s\tilde{\varphi}(s)ds\Bigg] + \lambda_0 \\
&\geq \gamma\rho_2\delta\Bigg[\int_0^1 s\phi^{-1}\left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right)ds + \lambda_0 \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\xi_i \int_0^1 s\phi^{-1}\left(\int_0^s q(r)dr + \frac{\boldsymbol{\sigma}(q)}{1-\sum_{i=1}^{m-2} b_i}\right)ds\Bigg] \\
&= \gamma\rho_2 + \lambda_0.
\end{aligned}
$$

This means that $\gamma\rho_2 \geq \gamma\rho_2 + \lambda_0$, which is a contradiction. Consequently, it follows from Theorem 2.1(A2) that

$$i(F, \Omega_{\rho_2}, K) = 0.$$

Lemma 3.1(ii) and the fact that $\rho_1 < \gamma\rho_2$ implies $\overline{K_{\rho_1}} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. From Theorem 2.1(A3), it follows that F has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K^*_{\rho_1}}$. Now $u_1 \notin K^*_{\rho_1}$ implies $u_1 < \rho_1\varphi(t)$ or $u_1 > \rho_1$. Moreover, $u_1 < \rho_1\varphi(t)$ implies $f^*(t,u) \neq f(t,u)$, so we have $u_1 > \rho_1$. Therefore, BVP (1)–(2) has a positive solution.

If condition (B2) holds, the proof is similar. This completes the proof of the theorem. □

The following results guaranteeing the existence of two positive solutions are analogous to those above and are proved in a similar fashion.

Theorem 3.2. *In addition to (H1)–(H3) assume that one of the following conditions holds:*

(C1) *There exist ρ_1, ρ_2, $\rho_3 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that*
  - (i) *$f(t, u(t)) > 0$ for $t \in [0, 1]$ and $u(t) \in [\rho_1\varphi(t), +\infty)$, and*
  - (ii) *$f^{\rho_1}_{\varphi(t)\rho_1} \leq \phi(\mu)$ for $t \in [0, 1]$, $f^{\rho_2}_{\gamma\rho_2} \geq \phi(\delta\gamma)$, and $u \neq Fu$ for all $u \in \partial\Omega\rho_2$, and $f^{\rho_3}_{\varphi(t)\rho_3} \leq \phi(\mu)$.*

(C2) *There exist ρ_1, ρ_2, $\rho_3 \in (0, +\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3$ such that*
  - (i) *$f(t, u(t)) > 0$ for $t \in [0, 1]$ and $u(t) \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, +\infty)$, and*

(ii) $f^{\rho_1}_{\gamma\rho_1} \geq \phi(\delta\gamma)$, $f^{\rho_2}_{\varphi(t)\rho_2} \leq \phi(\mu)$ *for* $t \in [0,1]$, $u \neq Fu$ *for all* $u \in \partial\Omega\rho_2$, *and* $f^{\rho_3}_{\gamma\rho_3} \geq \phi(\delta\gamma)$.

Then the BVP (1)– (2) *has two positive solutions.*

Corollary 3.1. *Assume that (H1)–(H3) hold and there exist* ρ', $\rho \in (0,+\infty)$ *with* $\rho' < \gamma\rho$ *such that one of the following assumptions holds:*

(D1) (i) $f(t,u(t)) > 0$ *for* $t \in [0,1]$ *and* $u(t) \in [\rho'\varphi(t),+\infty)$, *and*

(ii) $f^{\rho'}_{\varphi(t)\rho'} \leq \phi(\mu)$ *for* $t \in [0,1]$, $f^{\rho}_{\gamma\rho} \geq \phi(\delta\gamma)$, $u \neq Fu$ *for all* $u \in \partial\Omega\rho$, *and* $0 \leq f^{\infty} < \phi(\mu)$.

(D2) *There exist* ρ', $\rho \in (0,+\infty)$ *with* $\rho' < \rho$ *such that*

(i) $f(t,u(t)) > 0$ *for* $t \in [0,1]$ *and* $u(t) \in [\min\{\gamma\rho', \rho\varphi(t)\},+\infty)$, *and*

(ii) $f^{\rho'}_{\gamma\rho'} \geq \phi(\delta\gamma)$, $f^{\rho}_{\varphi(t)\rho} \leq \phi(\mu)$ *for* $t \in [0,1]$, $u \neq Fu$ *for all* $u \in \partial\Omega\rho$, *and* $\phi(\delta) < f_\infty \leq \infty$.

Then (1)–(2) *has two positive solutions.*

4. Application

As an example, consider the problem

$$(\phi(u''))' + f(t,u(t)) = 0, \quad t \in (0,1), \tag{16}$$

$$u(0) = \frac{1}{4}u(\frac{1}{3}), \quad \phi(u''(0)) = \frac{1}{2}\phi(u''(\frac{1}{3})), \quad u'(1) = 0, \tag{17}$$

where

$$\phi(u) = \begin{cases} -u^2, & u \leq 0, \\ u^2, & u > 0, \end{cases}$$

$$f(t,u) = \begin{cases} \frac{1}{3}(1+t)\left(u(t) - \frac{\varphi(t)}{2}\right)^{13}, & \text{for } 0 < u \leq 3, \\ \frac{1}{3}(1+t)\left(3 - \frac{\varphi(t)}{2}\right)^{13}, & \text{for } u > 3, \end{cases}$$

$f(t,0) \equiv 0$, and φ is defined in (11). It is easy to see that $f : [0,1]\times[0,+\infty) \to (-\infty,+\infty)$ is continuous. With $q(t) \equiv 1$, $\xi_1 = \frac{1}{3}$, $a_1 = \frac{1}{4}$, and $b_1 = \frac{1}{2}$, calculations show that

$$\begin{aligned} \gamma &= \frac{a_1\xi_1}{1-a_1(1-\xi_1)} = \frac{\frac{1}{4}\times\frac{1}{3}}{1-\frac{1}{4}(1-\frac{1}{3})} = \frac{1}{10}, \\ \frac{1}{\mu} &= \int_0^1 \phi^{-1}\left(s + \frac{b_1\xi_1}{1-b_1}\right)ds + \frac{1}{1-a_1}a_1\xi_1\int_0^1 \phi^{-1}\left(s + \frac{b_1\xi_1}{1-b_1}\right)ds \\ &= \int_0^1 \left(s+\frac{1}{3}\right)^{\frac{1}{2}} ds + \frac{1}{9}\int_0^1 \left(s+\frac{1}{3}\right)^{\frac{1}{2}} ds \approx 0.9979, \\ \frac{1}{\delta} &= \int_0^1 s\phi^{-1}\left(s + \frac{b_1\xi_1}{1-b_1}\right)ds + \frac{1}{1-a_1}a_1\xi_1\int_0^1 s\phi^{-1}\left(s + \frac{b_1\xi_1}{1-b_1}\right)ds \end{aligned}$$

$$= \int_0^1 s\left(s+\frac{1}{3}\right)^{\frac{1}{2}} ds + \frac{1}{9}\int_0^1 s\left(s+\frac{1}{3}\right)^{\frac{1}{2}} ds \approx 0.5512.$$

Hence, $\mu \approx 1.0021$ and $\delta \approx 1.8142$. Choosing $\rho_1 = 1$ and $\rho_2 = 20$, it is easy to see that $1 = \rho_1 < \gamma\rho_2 = \frac{1}{10} \times 20 = 2$, $f(t, u(t)) > 0$, for $t \in [0, 1]$ and $u(t) \in [\varphi(t), +\infty)$, and f satisfies

$$\begin{aligned} f^{\rho_1}_{\rho_1\varphi} &= \max\left\{\max_{t\in[0,1]} \frac{f(t,u)}{1^2} : u \in [\varphi(t)\rho_1, \rho_1]\right\} = \frac{2}{3} \approx 0.666667 \\ &< \phi(\mu) = \mu^2 = (1.0021)^2 = 1.0042, \\ f^{\rho_2}_{\gamma\rho_2} &= \min\left\{\min_{t\in[0,1]} \frac{f(t,u)}{20^2} : u \in [\gamma\rho_2, \rho_2]\right\} = \frac{5^{13}}{2^{13} \times 3 \times 20^2} \approx 124.176 \\ &> \phi(\delta\gamma) = (\delta\gamma)^2 = \left(1.8142 \times \frac{1}{10}\right)^2 \approx 0.0329. \end{aligned}$$

Thus, condition (B1) of Theorem 3.1 is satisfied, so the BVP (16)–(17) has at least one positive solution.

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References

[1] D. R. Anderson, *Green's function for a third-order generalized right focal problem*, J. Math. Anal. Appl. **288** (2003), 1–14.

[2] C. Bai and J. Fang, *Existence of multiple positive solutions for nonlinear m-point boundary value problems*, J. Math. Anal. Appl. **81** (2003), 76–85.

[3] M. Benchohra, J. J. Nieto, and A. Ouahab, *Second-order boundary value problem with integral boundary conditions*, Bound. Value Probl. **2011** (2011), 1–9.

[4] A. Boucherif, *Second-order boundary value problems with integral boundary conditions*, Nonlinear Anal. **70** (2009), 364–371.

[5] A. Dogan, *The existence of positive solutions for a semipositone second-order m-point boundary value problem*, Dynam. Syst. Appl. **24** (2015), 419–428.

[6] A. Dogan, *Eigenvalue problems for nonlinear third-order m-point p-Laplacian dynamic equations on time scales*, Math. Meth. Appl. Sci. **39** (2016), 1634–1645.

[7] J. R. Graef and L. Kong, *Positive solutions for a class of higher order boundary value problems with fractional q-derivatives*, Appl. Math. Comput. **218** (2012), 9682–9689.

[8] J. R. Graef and L. Kong, *Positive solutions for third order semipositone boundary value problems*, Appl. Math. Lett. **22** (2009), 1154–1160.

[9] J. R. Graef, L. Kong, Q. Kong, and M. Wang, *Fractional boundary value problems with integral boundary conditions*, Appl. Anal. **92** (2013), 2008–2020.

[10] M. Gregus, *Third Order Linear Differential Equations*, Math. and its Appl., Reidel, Dordrecht, 1987.

[11] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.
[12] J. Henderson and N. Kosmatov, *Three-point third-order problems with sign changing nonlinear term*, Electron. J. Differential Equations, **2014** (2014), No. 175, pp. 1–10.
[13] J. Henderson and R. Luca, *Positive solutions for singular systems of multi-point boundary value problems*, Math. Meth. Appl. Sci. **36** (2013), 814–828.
[14] K. Q. Lan, *Multiple positive solutions of semilinear differential equations with singularities*, J. London Math. Soc. **63** (2001), 690–704.
[15] S. H. Li, *Positive solutions of nonlinear singular third-order two-point boundary value problem*, J. Math. Anal. Appl. **323** (2006), 413–425.
[16] Z. Liu, L. Debnath, and S. M. Kang, *Existence of monotone positive solutions to a third order two-point generalized right focal boundary value problem*, Comput. Math. Appl. **55** (2008), 356–367.
[17] Z. Liu, J. S. Ume, D. R. Anderson, and S. M. Kang, *Twin monotone positive solutions to a singular nonlinear third-order differential equation*, J. Math. Anal. Appl. **334** (2007), 299–313.
[18] D. Ma, Z. Du, and W. Ge, *Existence and iteration of monotone positive solutions for multipoint boundary value problem with p- Laplacian operator*, Comput. Math. Appl. **50** (2005), 729–739.
[19] R. Ma, *Positive solutions of nonlinear m-point boundary value problems*, Comput. Math. Appl. **42** (2001), 755–765.
[20] F. M. Minhós, *On some third order nonlinear boundary value problems: Existence, location and multiplicity results*, J. Math. Anal. Appl. **339** (2008), 1342–1353.
[21] Y. Sang and H. Su, *Positive solutions of nonlinear third-order m-point BVP for an increasing homeomorphism and homomorphism with sign-changing nonlinearity*, J. Comput. Appl. Math. **225** (2009), 288–300.
[22] J. P. Sun and J. Zhao, *Iterative technique for a third-order three-point BVP with sign-changing Green's function*, Electron. J. Differential Equations, **2013** (2013), No. 215, pp. 1–9.
[23] Y. Sun and L. Liu, *Solvability for a nonlinear second-order three-point boundary value problem*, J. Math. Anal. Appl. **296** (2004), 265–275.
[24] Y. Wang and W. Ge, *Positive solutions for multipoint boundary value problems with one-dimensional p-Laplacian*, Nonlinear Anal. **66** (6) (2007), 1246–1256.
[25] Y. Wang and C. Hou, *Existence of multiple positive solutions for one-dimensional p-Laplacian*, J. Math. Anal. Appl. **315** (2006), 144–153.
[26] C. Yang and J. Yan, *Positive solutions for third-order Sturm-Liouville boundary value problems with p-Laplacian*, Comput. Math. Appl. **59** (2010), 2059–2066.
[27] X. Zhang, L. Liu, and C. Wu, *Nontrivial solution of third-order nonlinear eigenvalue problems*, Appl. Math. Comput. **176** (2006), 714–721.
[28] C. L. Zhou and D. X. Ma, *Existence and iteration of positive solutions for a generalized right-focal boundary value problem with p-Laplacian operator*, J. Math. Anal. Appl. **324** (2006), 409–424.

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